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MESSENGER OF MATHEMATICS.

RELATIONS CONNECTING QUANTITIES OF THE FORM $1+2^{-n}+3^{-n}+4^{-n}+$ &c.

By *J. W. L. Glaisher.*

§ 1. In the course of an investigation upon which I am engaged I have met with several systems of relations connecting S_2, S_4, &c., and S_3, S_5, &c., where

$$S_n = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \&c.$$

These relations, which were new to me, are stated in the present paper.

Single series involving S_2, S_4, &c., § 2.

§ 2. The following is a system of relations connecting S_2, S_4, &c.:—

$$\frac{S_2}{3.4} + \frac{S_4}{5.6} + \frac{S_6}{7.8} + \&c. = \tfrac{1}{2}\,\frac{1}{1.2},$$

$$\frac{S_2}{4.5.6} + \frac{S_4}{6.7.8} + \frac{S_6}{8.9.10} + \&c. = \tfrac{1}{2}\,\frac{1}{2.3.4},$$

$$\frac{S_2}{5.6.7.8} + \frac{S_4}{7.8.9.10} + \&c. = \tfrac{1}{2}\,\frac{1}{3.4.5.6},$$

$$\frac{S_2}{6.7\ldots10} + \frac{S_4}{8.9\ldots12} + \&c. = \tfrac{1}{2}\,\frac{1}{4.5\ldots8},$$

&c. &c.,

the general formula being

$$\sum_{r=1} \frac{S_{2r}}{(2r+k)(2r+k+1)\ldots(2r+2k)} = \tfrac{1}{2}\,\frac{1}{k(k+1)\ldots2k},$$

where k is any positive integer greater than zero.

Single series involving S_3, S_5, &c., § 3.

§ 3. The following is a system of relations connecting S_3, S_5, &c.:—

$$\frac{S_3}{4.5.6}+\frac{2S_5}{6.7\,8}+\frac{3S_7}{8.9.10}+\&c.=\tfrac{1}{4}\frac{1}{1.2.3},$$

$$\frac{S_3}{5.6.7.8}+\frac{2S_5}{7.8.9.10}+\frac{3S_7}{9.10.11.12}+\&c.=\tfrac{1}{4}\frac{1}{2.3.4.5},$$

$$\frac{S_3}{6.7\ldots10}+\frac{2S_5}{8.9\ldots12}+\frac{3S_7}{10.11\ldots14}+\&c.=\tfrac{1}{4}\frac{1}{3.4\ldots7},$$

&c. &c.,

the general formula being

$$\sum_{r=1}\frac{rS_{2r+1}}{(2r+k+1)(2r+k+2)\ldots(2r+2k+2)}=\tfrac{1}{4}\frac{1}{k(k+1)\ldots(2k+1)},$$

where k is any positive integer greater than zero.

Relations between series involving S_2, S_4, &c., § 4.

§ 4. In the following relations a series of the form

$$\frac{S_2}{(k+2)(k+3)}+\frac{S_4}{(k+4)(k+5)}+\frac{S_6}{(k+6)(k+7)}+\&c.$$

(k being any positive integer) is linearly connected with one of the form

$$\frac{S_2}{3.4.5\ldots(k+3)}+\frac{S_4}{5.6.7\ldots(k+5)}+\frac{S_6}{7.8.9\ldots(k+7)}+\&c.$$

In the first series the denominators contain two factors, in the second series they contain $k+1$ factors.

The relations are

$$\frac{S_2}{4.5}+\frac{S_4}{6.7}+\frac{S_6}{8.9}+\&c.=\tfrac{1}{4}-2!\left\{\frac{S_2}{3.4.5}+\frac{S_4}{5.6.7}+\frac{S_6}{7.8.9}+\&c.\right\}.$$

$$\frac{S_2}{5.6}+\frac{S_4}{7.8}+\frac{S_6}{9.10}+\&c.=\tfrac{1}{6}-3!\left\{\frac{S_2}{3.4.5.6}+\frac{S_4}{5.6.7.8}+\&c.\right\}.$$

$$\frac{S_2}{6.7}+\frac{S_4}{8.9}+\frac{S_6}{10.11}+\&c.=\tfrac{1}{8}-4!\left\{\frac{S_2}{3.4.5.6.7}+\frac{S_4}{5.6.7.8.9}+\&c.\right\},$$

&c. &c.,

the general formula being

$$\sum_{r=1} \frac{S_{2r}}{(2r+k)(2r+k+1)}$$

$$= \frac{1}{2k} - k! \sum_{r=1} \frac{S_{2r}}{(2r+1)(2r+2)\ldots(2r+k+1)},$$

where k is any positive integer.

The case $k=1$ is equivalent to the first formula of § 2.

By combining the two series in the first of the formulæ written above (corresponding to $k=2$), we again obtain the first formula of § 2.

Relations between series involving S_3, S_5, &c., § 5.

§ 5. The corresponding relations involving S_3, S_5, &c. connect linearly the series

$$\frac{S_3}{(k+3)(k+4)} + \frac{S_5}{(k+5)(k+6)} + \frac{S_7}{(k+7)(k+8)} + \text{&c.}$$

(k being any positive integer) with the series

$$\frac{S_3}{4.5\ldots(k+4)} + \frac{S_5}{6.7\ldots(k+6)} + \frac{S_7}{8.9\ldots(k+8)} + \text{&c.},$$

in which each term contains $k+1$ factors.

The relations are

$$\frac{S_3}{5.6} + \frac{S_5}{7.8} + \frac{S_7}{9.10} + \text{&c.} = \tfrac{1}{2}\frac{1}{2.3} + 2!\left\{\frac{S_3}{4.5.6} + \frac{S_5}{6.7.8} + \frac{S_7}{8.9.10} + \text{&c.}\right\},$$

$$\frac{S_3}{6.7} + \frac{S_5}{8.9} + \frac{S_7}{10.11} + \text{&c.} = \tfrac{1}{2}\frac{2}{3.4} + 3!\left\{\frac{S_3}{4.5\,6.7} + \frac{S_5}{6.7.8.9} + \text{&c.}\right\},$$

$$\frac{S_3}{7.8} + \frac{S_5}{9.10} + \frac{S_7}{11.12} + \text{&c.} = \tfrac{1}{2}\frac{3}{4.5} + 4!\left\{\frac{S_4}{4.5\ldots8} + \frac{S_5}{6.7\ldots10} + \text{&c.}\right\},$$

&c. &c.,

the general formula being

$$\sum_{r=1} \frac{S_{2r+1}}{(2r+k+1)(2r+k+2)}$$

$$= \tfrac{1}{2}\frac{k-1}{k(k+1)} + k! \sum_{r=1} \frac{S_{2r+1}}{(2r+2)(2r+3)\ldots(2r+k+2)},$$

where k is any positive integer.

The case $k=1$ gives an identity. By combining the two series in the first formula (corresponding to $k=2$), we obtain the first formula of § 3.

Other relations between series involving S_2, S_4, &c., §§ 6—10.

§ 6. By subtraction from the results in § 4 it follows that

$$2\left\{\frac{S_2}{4.5.6}+\frac{S_4}{6.7.8}+\&c.\right\}=\tfrac{1}{2}\frac{1}{2.3}-2!\left\{\frac{S_2}{4.5.6}+\frac{S_4}{6.7.8}+\&c.\right\},$$

$$2\left\{\frac{S_2}{5.6.7}+\frac{S_4}{7.8.9}+\&c.\right\}=\tfrac{1}{2}\frac{1}{3.4}-3!\left\{\frac{S_2}{4.5.6.7}+\frac{S_4}{6.7.8.9}+\&c.\right\},$$

$$2\left\{\frac{S_2}{6.7.8}+\frac{S_4}{8.9.10}+\&c.\right\}=\tfrac{1}{2}\frac{1}{4.5}-4!\left\{\frac{S_2}{4.5.6.7.8}+\frac{S_4}{6.7.8.9.10}+\&c.\right\},$$

&c. &c.

The first of these formulæ is equivalent to the second formula in § 2, as also is the second when the series are combined.

The general formula is

$$2\sum_{r=1}\frac{S_{2r}}{(2r+k)(2r+k+1)(2r+k+2)}$$

$$=\tfrac{1}{2}\frac{1}{k(k+1)}-k!\sum_{r=1}\frac{S_{2r}}{(2r+2)(2r+3)\ldots(2r+k+2)},$$

where k is any positive integer. For $k=1$ the formula is equivalent to the first result in § 4.

§ 7. Similarly, by subtraction from § 6, we find

$$3!\left\{\frac{S_2}{4.5.6.7}+\frac{S_4}{6.7.8.9}+\&c.\right\}$$
$$=\tfrac{1}{2}\frac{2}{2.3.4}-2!\left\{\frac{S_2}{5.6.7}+\frac{S_4}{7.8.9}+\&c.\right\},$$

$$3!\left\{\frac{S_2}{5.6.7.8}+\frac{S_4}{7.8.9.10}+\&c.\right\}$$
$$=\tfrac{1}{2}\frac{2}{3\,4.5}-3!\left\{\frac{S_2}{5.6.7.8}+\frac{S_4}{7.8.9.10}+\&c.\right\},$$

$$3!\left\{\frac{S_2}{6.7.8.9}+\frac{S_4}{8.9.10.11}+\&c.\right\}$$
$$=\tfrac{1}{2}\frac{2}{4.5.6}-4!\left\{\frac{S_2}{5.6.7.8.9}+\frac{S_4}{7.8.9.10.11}+\&c.\right\},$$

&c. &c.

The first formula is the same as the second of § 6; and the second formula is equivalent to the third of § 2, as also is the third formula when the series are combined.

The general formula is

$$3!\sum_{r=1}\frac{S_{2r}}{(2r+k)(2r+k+1)(2r+k+2)(2r+k+3)}$$

$$=\tfrac{1}{2}\frac{2!}{k(k+1)(k+2)}-k!\sum_{r=1}\frac{S_{2r}}{(2r+3)(2r+4)\ldots(2r+k+3)},$$

where k is any positive integer. For $k=1$ the formula is equivalent to the second result in § 4.

§ 8. Similarly, by continued subtraction, omitting formulæ already given, we find

$$4!\left\{\frac{S_2}{7.8\ldots11}+\frac{S_4}{9.10\ldots13}+\&c\right\}$$

$$=\tfrac{1}{2}\frac{3!}{5.6.7.8}-5!\left\{\frac{S_2}{6.7\ldots11}+\frac{S_4}{8.9\ldots13}+\&c.\right\},$$

$$4!\left\{\frac{S_2}{8.9\ldots12}+\frac{S_4}{10.11\ldots14}+\&c.\right\}$$

$$=\tfrac{1}{2}\frac{3!}{6.7.8.9}-5!\left\{\frac{S_2}{6.7\ldots12}+\frac{S_4}{8.9\ldots14}+\&c.\right\}.$$

&c. &c.;

$$5!\left\{\frac{S_2}{8.9\ldots13}+\frac{S_4}{10.11\ldots13}+\&c.\right\}$$

$$=\tfrac{1}{2}\frac{4!}{6.7\ldots10}-6!\left\{\frac{S_2}{7.8\ldots13}+\frac{S_4}{7.8\ldots15}+\&c.\right\},$$

&c. &c.;

$$6!\left\{\frac{S_2}{9.10\ldots15}+\frac{S_4}{11.12\ldots17}+\&c.\right\}$$

$$=\tfrac{1}{2}\frac{5!}{7.8\ldots12}-7!\left\{\frac{S_2}{8.9\ldots15}+\frac{S_4}{8.9\ldots17}+\&c.\right\},$$

&c. &c.,

&c. &c.

§ 9. The general formula which includes all the results (relating to S_2, S_4, &c.) in §§ 4 and 6—8 is

$$i!\sum_{r=1}\frac{S_{2r}}{(2r+k)(2r+k+1)\ldots(2r+k+i)}$$
$$=\tfrac{1}{2}\frac{(i-1)!}{k(k+1)(k+i-1)}-k!\sum_{r=1}\frac{S_{2r}}{(2r+i)(2r+i+1)\ldots(2r+k+i)},$$

where k and i are any positive integers. The number of factors in the denominators on the left-hand side is $i+1$, and on the right-hand side is $k+1$.

The numbers i and k are involved symmetrically. The quantity not subject to the summation sign must therefore be symmetrical, as is the case, since it may be written

$$\tfrac{1}{2}\frac{(k-1)!\,(i-1)!}{(k+i-1)!}.$$

§ 10. Putting the two series on the same side of the equation, the relation becomes

$$i!\sum_{r=1}\frac{S_{2r}}{(2r+k)(2r+k+1)\ldots(2r+k+i)}$$
$$+k!\sum_{r=1}\frac{S_{2r}}{(2r+i)(2r+i+1)\ldots(2r+i+k)}=\tfrac{1}{2}\frac{(k-1)!\,(i-1)!}{(k+i-1)!}.$$

Other relations between series involving S_3, S_5, &c., §§ 11—14.

§ 11. The corresponding formulæ relating to S_3, S_5, &c., which are derivable from § 5, are

$$2\left\{\frac{S_3}{6.7.8}+\frac{S_5}{8.9.10}+\&c.\right\}$$
$$=\tfrac{1}{2}\frac{1}{3.4.5}+3!\left\{\frac{S_3}{5.6.7.8}+\frac{S_5}{7.8.9.10}+\&c.\right\},$$

$$2\left\{\frac{S_3}{7.8.9}+\frac{S_5}{9.10.11}+\&c.\right\}$$
$$=\tfrac{1}{2}\frac{2}{4.5.6}+4!\left\{\frac{S_3}{5.6.7.8.9}+\frac{S_5}{7.8.9.10.11}+\&c.\right\},$$

$$2\left\{\frac{S_3}{8.9.10}+\frac{S_5}{10.11.12}+\&c.\right\}$$
$$=\tfrac{1}{2}\frac{3}{5.6.7}+5!\left\{\frac{S_3}{5.6\ldots10}+\frac{S_5}{7.8\ldots12}+\&c.\right\},$$

&c. &c.;

$$3!\left\{\frac{S_3}{7.8.9.10}+\frac{S_5}{9.10.11.12}+\text{&c.}\right\}$$
$$=\frac{1}{4.5.6.7}+4!\left\{\frac{S_3}{6.7.8.9.10}+\frac{S_5}{8.9.10.11.12}+\text{&c.}\right\},$$

$$3!\left\{\frac{S_3}{8.9.10.11}+\frac{S_5}{10.11.12.13}+\text{&c.}\right\}$$
$$=\frac{2}{5.6.7.8}+5!\left\{\frac{S_3}{6.7\ldots11}+\frac{S_5}{8.9\ldots13}+\text{&c.}\right\},$$
$$\text{&c.}\qquad\qquad\text{&c.};$$

$$4!\left\{\frac{S_3}{8.9\ldots12}+\frac{S_5}{10.11\ldots14}+\text{&c.}\right\}$$
$$=3\,\frac{1}{5.6\ldots9}+5!\left\{\frac{S_3}{7.8\ldots12}+\frac{S_5}{9.10\ldots14}+\text{&c.}\right\},$$
$$\text{&c.}\qquad\qquad\text{&c.},$$

the general formulæ for the cases when there are 3, 4, 5, ... factors in the denominators on the left-hand side being

$$2\sum_{r=1}\frac{S_{2r+1}}{(2r+k+1)(2r+k+2)(2r+k+3)}$$
$$=\tfrac{1}{2}\,\frac{k-2}{k(k+1)(k+2)}+k!\sum_{r=1}\frac{S_{2r+1}}{(2r+3)(2r+4)\ldots(2r+k+3)},$$

$$3!\sum_{r=1}\frac{S_{2r+1}}{(2r+k+1)\ldots(2r+k+4)}$$
$$=\frac{2!}{2}\,\frac{k-3}{k(k+1)\ldots(k+3)}+k!\sum_{r=1}\frac{S_{2r+1}}{(2r+4)\ldots(2r+k+4)},$$

$$4!\sum_{r=1}\frac{S_{2r+1}}{(2r+k+1)\ldots(2r+k+5)}$$
$$=\frac{3!}{2}\,\frac{k-4}{k(k+1)\ldots(k+4)}+k!\sum_{r=1}\frac{S_{2r+1}}{(2r+5)\ldots(2r+k+5)},$$
$$\text{&c.}\qquad\qquad\text{&c.}$$

§ 12. The more general formula which includes all these results, and which corresponds to the general formula relating to S_2, S_4, &c. in § 9, is

$$i!\sum_{r=1}\frac{S_{2r+1}}{(2r+k+1)(2r+k+2)\ldots(2r+k+i+1)}$$
$$=\frac{(i-1)!}{2}\frac{k-i}{k(k+1)\ldots(k+i)}$$
$$+k!\sum_{r=1}\frac{S_{2r+1}}{(2r+i+1)(2r+i+2)\ldots(2r+i+k+1)},$$

where k and i are any positive integers.

§ 13. The numbers k and i are involved symmetrically, and this formula may be written

$$i!\sum_{r=1}\frac{S_{2r+1}}{(2r+k+1)(2r+k+2)\ldots(2r+k+i+1)}$$
$$-k!\sum_{r=1}\frac{S_{2r+1}}{(2r+i+1)(2r+i+2)\ldots(2r+i+k+1)}=\frac{k-i}{2}\frac{(k-1)!\,(i-1)!}{(k+1)!}.$$

§ 14. It may be remarked that by adding the formulæ in §§ 10 and 13, we obtain the following relation, in which all the S's (*i.e.*, with both even and uneven suffixes) occur,

$$i!\sum_{r=2}\frac{S_r}{(r+k)(r+k+1)\ldots(r+k+i)}$$
$$+k!\sum_{r=2}(-1)^r\frac{S_r}{(r+i)(r+i+1)\ldots(r+i+k)}=\tfrac{1}{2}\frac{k!\,(i-1)!}{(k+i)!}.$$

The same result is also obtained if the formulæ are combined by subtraction.

Proofs of the formulæ, §§ 15—16.

§ 15. I obtained the formulæ in §§ 4 and 5 by deriving them from the actual values of the series involved, which were found to satisfy the relations given, *e.g.*, the formulæ were derived from the values of the series

$$\frac{S_2}{(k+2)(k+3)}+\frac{S_4}{(k+4)(k+5)}+\&c.,$$

and

$$\frac{S_2}{3.4\ldots(k+3)}+\frac{S_4}{5.6\ldots(k+5)}+\&c.,$$

and those of the corresponding series in § 5. The values of the former series were expressed in terms of α_0, α_2, ..., and those of the latter in terms of α_1, α_3, ..., where the quantities

α_n may be defined as the constants occurring in the formula

$$1^n \log 1 + 2^n \log 2 + \ldots + x^n \log x$$

$$= \alpha_n + \left\{\frac{x^{n+1}}{n+1} + \tfrac{1}{2}x^n + \frac{B_1}{2!}nx^{n-1} + \ldots\right\}\log x - \frac{x^{n+1}}{(n+1)^2} + \frac{B_1}{2!}x^{n-1} - \&c.^*$$

§ 16. The results in §§ 6—13, though originally obtained independently, are deducible from those in §§ 4 and 5, as has been shown in this paper. These results include those in § 2, which, however, were first obtained by subtraction from pairs of series whose sums had been obtained and found to be equal. The formulæ in § 3 were also obtained in this manner, *i.e.*, by subtraction from pairs of series whose sums were equal.

It should be possible to give direct proofs of all the relations in this paper without first summing the series.

Series for Euler's constant, §§ 17—19.

§ 17. In the course of the investigation referred to in § 1 I have obtained also the following series for Euler's constant γ:

$$\gamma = \tfrac{5}{6} - 12\left\{\frac{S_3}{3.5.6} + \frac{S_5}{5.7.8} + \frac{S_7}{7.9.10} + \&c.\right\},$$

$$\gamma = \tfrac{47}{60} - 120\left\{\frac{S_3}{3.6.7.8} + \frac{S_5}{5.8.9.10} + \frac{S_7}{7.10.11.12} + \&c.\right\},$$

$$\gamma = \tfrac{319}{420} - 1680\left\{\frac{S_3}{3.7.8.9.10} + \frac{S_5}{5.9.10.11.12} + \frac{S_7}{7.11.12.13.14} + \&c.\right\}.$$

In the denominators a gap occurs after the first factor 3, 5, 7, ...

These formulæ were obtained in the same manner as those given in the preceding portion of this paper, viz., by subtraction from series whose values had been expressed in terms of γ and $\alpha_1, \alpha_3, \ldots$.†

§ 18. The formula

$$\gamma = 1 - 2\left\{\frac{S_3}{3.4} + \frac{S_5}{5.6} + \frac{S_7}{7.8} + \&c.\right\}\ddagger$$

---

* This formula is given at full length in vol. xxviii, p. 109, of the *Quarterly Journal*.

† The third series had to be specially sought for, as in the class of formulæ with which I was concerned I found that, in general, when the α's disappeared by subtraction so also did γ.

‡ *Quarterly Journal*, vol. xxviii., p. 59.

belongs to the same system as the three just given; and the four formulæ seem to point to the existence of a general formula for γ of the form

$$\gamma = \lambda_r - \mu_r \left\{ \frac{S_3}{3(r+3)(r+4)\ldots(2r+2)} + \frac{S_5}{5(r+5)(r+6)\ldots(2r+4)} + \&c. \right\},$$

where r is any positive integer, λ_r is a fraction, and μ_r an integer which presumably $= (r+1)(r+2)\ldots 2r$.

§ 19. As is of frequent occurrence in formulæ involving S's, the constant γ seems to correspond to S_1, which cannot occur as it is infinite. Thus the formulæ may be written

$$\frac{\gamma}{1.2} + \frac{S_3}{3.4} + \frac{S_5}{5.6} + \frac{S_7}{7.8} + \&c. = \frac{1}{2},$$

$$\frac{\gamma}{1.3.4} + \frac{S_3}{3.5.6} + \frac{S_5}{5.7.8} + \&c. = \frac{5}{72},$$

$$\frac{\gamma}{1.4.5.6} + \frac{S_3}{3.6.7.8} + \frac{S_5}{5.8.9.10} + \&c. = \frac{47}{7200},$$

$$\frac{\gamma}{1.5.6.7.8} + \frac{S_3}{3.7.8.9.10} + \frac{S_5}{5.9.10.11.12} + \&c. = \frac{319}{705600},$$

the general formula presumably being

$$\frac{\gamma}{1(r+1)(r+2)\ldots 2r} + \frac{S_3}{3(r+3)(r+4)\ldots(2r+2)} + \frac{S_5}{5(r+5)(r+6)\ldots(2r+4)} + \&c. = \lambda_r',$$

where λ_r' is a proper fraction.

SOME DEFINITE INTEGRALS.

By *S. Ramanujan.*

1. Consider the integral

$$\int_0^\infty \frac{\cos 2mx\, dx}{\{1+x^2/a^2\}\{1+x^2/(a+1)^2\}\{1+x^2/(a+2)^2\}\ldots},$$

where m and a are positive.

It can be easily proved that

$$\left\{1-\left(\frac{t}{a}\right)^2\right\}\left\{1-\left(\frac{t}{a+1}\right)^2\right\}\left\{1-\left(\frac{t}{a+2}\right)^2\right\}\ldots\left\{1-\left(\frac{t}{a+n-1}\right)^2\right\}$$
$$=\frac{\Gamma(a+n-t)\,\Gamma(a+n+t)\,\{\Gamma(a)\}^2}{\Gamma(a-t)\,\Gamma(a+t)\,\{\Gamma(a+n)\}^2},$$

where n is any positive integer. Hence, by splitting

$$\frac{1}{\{1+x^2/a^2\}\{1+x^2/(a+1)^2\}\ldots\{1+x^2/(a+n-1)^2\}}$$

into partial fractions, we see that it is equal to

$$\frac{2\Gamma(2a)\,\{\Gamma(a+n)\}^2}{\{\Gamma(a)\}^2\Gamma(n)\,\Gamma(2a+n)}\left\{\frac{a}{a^2+x^2}-\frac{2a}{1!}\,\frac{n-1}{n+2a}\,\frac{a+1}{(a+1)^2+x^2}\right.$$
$$\left.+\frac{2a\,(2a+1)}{2!}\,\frac{(n-1)\,(n-2)}{(n+2a)\,(n+2a+1)}\,\frac{a+2}{(a+2)^2+x^2}-\ldots\right\}.$$

Multiplying both sides by $\cos 2mx$ and integrating from 0 to ∞ with respect to x, we have

$$\int_0^\infty\frac{\cos 2mx\,dx}{\{1+x^2/a^2\}\{1+x^2/(a+1)^2\}\ldots\{1+x^2/(a+n-1)^2\}}$$
$$=\frac{\pi\,\Gamma(2a)\,\{\Gamma(a+n)\}^2}{\{\Gamma(a)\}^2\Gamma(n)\,\Gamma(2a+n)}\left\{e^{-2am}-\frac{2a}{1!}\,\frac{n-1}{n+2a}\,e^{-2(a+1)m}+\ldots\right\}.$$

The limit of the right-hand side, as $n\to\infty$, is

$$\frac{\pi\,\Gamma(2a)}{\{\Gamma(a)\}^2}\left\{e^{-2am}-\frac{2a}{1!}\,e^{-2(a+1)m}+\frac{2a\,(2a+1)}{2!}\,e^{-2(a+2)m}-\ldots\right\}$$
$$=\tfrac{1}{2}\sqrt{\pi}\,\frac{\Gamma(a+\tfrac{1}{2})}{\Gamma(a)}\,\mathrm{sech}^{2a}m.$$

Hence

$$(1)\quad\int_0^\infty\frac{\cos 2mx\,dx}{\{1+x^2/a^2\}\{1+x^2/(a+1)^2\}\ldots}=\tfrac{1}{2}\sqrt{\pi}\,\frac{\Gamma(a+\tfrac{1}{2})}{\Gamma(a)}\,\mathrm{sech}^{2a}m.$$

Since

$$\left\{1+\left(\frac{x}{a}\right)^2\right\}\left\{1+\left(\frac{x}{a+1}\right)^2\right\}\left\{1+\left(\frac{x}{a+2}\right)^2\right\}\ldots$$
$$=\frac{\{\Gamma(a)\}^2}{\Gamma(a+ix)\,\Gamma(a-ix)},$$

the formula (1) is equivalent to

$$(2)\quad \int_0^\infty |\Gamma(a+ix)|^2 \cos 2mx\, dx = \tfrac{1}{2}\sqrt{\pi}\,\Gamma(a)\,\Gamma(a+\tfrac{1}{2})\,\mathrm{sech}^{2a} m.$$

2. In a similar manner we can prove that

$$\int_0^\infty \frac{1+x^2/b^2}{1+x^2/a^2}\,\frac{1+x^2/(b+1)^2}{1+x^2/(a+1)^2}\,\frac{1+x^2/(b+2)^2}{1+x^2/(a+2)^2}\ldots \cos mx\, dx$$

$$= \frac{\pi\,\Gamma(2a)\,\{\Gamma(b)\}^2}{\{\Gamma(a)\}^2\,\Gamma(b+a)\,\Gamma(b-a)}\left\{e^{-am} - \frac{2a}{1!}\,\frac{b-a-1}{b+a}\,e^{-(a+1)m}\right.$$

$$\left. + \frac{2a(2a+1)}{2!}\,\frac{(b-a-1)(b-a-2)}{(b+a)(b+a+1)}\,e^{-(a+2)m} - \ldots\right\},$$

where m is positive and $0<a<b$. When $0<a<b-\frac{1}{2}$, the integral and the series remain convergent for $m=0$, and we obtain the formulæ

$$(3)\quad \int_0^\infty \frac{1+x^2/b^2}{1+x^2/a^2}\,\frac{1+x^2/(b+1)^2}{1+x^2/(a+1)^2}\,\frac{1+x^2/(b+2)^2}{1+x^2/(a+2)^2}\ldots dx$$

$$= \tfrac{1}{2}\sqrt{\pi}\,\frac{\Gamma(a+\frac{1}{2})\,\Gamma(b)\,\Gamma(b-a-\frac{1}{2})}{\Gamma(a)\,\Gamma(b-\frac{1}{2})\,\Gamma(b-a)},$$

$$(4)\quad \int_0^\infty \left|\frac{\Gamma(a+ix)}{\Gamma(b+ix)}\right|^2 dx = \tfrac{1}{2}\sqrt{\pi}\,\frac{\Gamma(a)\,\Gamma(a+\frac{1}{2})\,\Gamma(b-a-\frac{1}{2})}{\Gamma(b-\frac{1}{2})\,\Gamma(b)\,\Gamma(b-a)}.$$

If $a_1, a_2, a_3, \ldots, a_n$ be n positive numbers in arithmetical progression, then

$$\int_0^\infty \frac{dx}{(a_1^2+x^2)(a_2^2+x^2)(a_3^2+x^2)\ldots(a_n^2+x^2)}$$

is a particular case of the above integral, and its value can be written down at once. Thus, for example, by putting $a=\frac{11}{10}$ and $b=\frac{61}{10}$, we obtain

$$\int_0^\infty \frac{dx}{(x^2+11^2)(x^2+21^2)(x^2+31^2)(x^2+41^2)(x^2+51^2)}$$

$$= \frac{5\pi}{12.13.16.17.18.22.23.24\ 31.32\ 41}.$$

3. It follows at once from equation (1), by applying Fourier's theorem

$$\int_0^\infty \cos ny\, dy \int_0^\infty \phi(x)\cos xy\, dx = \tfrac{1}{2}\pi\,\phi(n),$$

that, when a and n are positive,

$$(5) \qquad \int_0^\infty \operatorname{sech}^{2a} x \cos 2nx \, dx$$

$$= \tfrac{1}{2}\sqrt{\pi}\,\frac{\Gamma(a)}{\Gamma(a+\tfrac{1}{2})}\,\frac{1}{\{1+n^2/a^2\}\{1+n^2/(a+1)^2\}\{1+n^2/(a+2)^2\}\ldots}$$

$$= \tfrac{1}{2}\sqrt{\pi}\,\frac{|\Gamma(a+in)|^2}{\Gamma(a)\,\Gamma(a+\tfrac{1}{2})}.$$

Hence the function

$$\phi(a) = \int_0^\infty \operatorname{sech}^a x \cos nx \, dx \quad (0 < a < 2)$$

satisfies the functional equation

$$\phi(a)\,\phi(2-a) = \frac{\pi \sin \pi a}{2(1-a)(\cosh \pi n - \cos \pi a)}.$$

4. Let $$\int_a^b f(x)\,F(nx)\,dx = \psi(n),$$

and $$\int_a^\beta \phi(x)\,F(nx)\,dx = \chi(n).$$

Then, if we suppose the functions f, ϕ, and F to be such that the order of integration is indifferent, we have

$$(6) \quad \int_a^b f(x)\,\chi(nx)\,dx = \int_a^\beta dy \int_a^b f(x)\,\phi(y)\,F(nxy)\,dx$$

$$= \int_a^\beta \phi(y)\,\psi(ny)\,dy.$$

A number of curious relations between definite integrals may be deduced from this result. We have, for example, the formulæ

$$(7) \qquad \int_0^\infty \frac{\cos 2nx}{\cosh \pi x}\,dx = \frac{1}{2\cosh n},$$

$$(8) \qquad \int_0^\infty \frac{\cos 2nx\,dx}{1+2\cosh \tfrac{2}{3}\pi x} = \frac{\sqrt{3}}{2(1+2\cosh 2n)},$$

$$(9) \qquad \int_0^\infty e^{-x^2}\cos 2nx\,dx = \tfrac{1}{2}\sqrt{\pi}\,e^{-n^2}.$$

By applying the general result (6) to the integrals (7) and (8), we obtain

$$\sqrt{3}\int_0^\infty \frac{dx}{\cosh \pi x\,(1+2\cosh 2nx)} = \int_0^\infty \frac{dx}{\cosh nx\,(1+2\cosh \tfrac{2}{3}\pi x)};$$

or, in other words, if $\alpha\beta = \frac{3}{4}\pi^2$, then

$$(10) \qquad \sqrt{\alpha}\int_0^\infty \frac{dx}{\cosh\alpha x\,(1+2\cosh\pi x)} = \sqrt{\beta}\int_0^\infty \frac{dx}{\cosh\beta x\,(1+2\cosh\pi x)}.$$

In the same way, from (8) and (9), we obtain

$$(11) \qquad \sqrt{\alpha}\int_0^\infty \frac{e^{-x^2}\,dx}{1+2\cosh\alpha x} = \sqrt{\beta}\int_0^\infty \frac{e^{-x^2}\,dx}{1+2\cosh\beta x},$$

with the condition $\alpha\beta = \frac{4}{3}\pi$; and, from (7) and (9),

$$(12) \qquad \sqrt{\alpha}\int_0^\infty \frac{e^{-x^2}}{\cosh\alpha x}\,dx = \sqrt{\beta}\int_0^\infty \frac{e^{-x^2}}{\cosh\beta x}\,dx,$$

with the condition $\alpha\beta = \pi$.*

Similarly, by taking the two integrals

$$\int_0^\infty \frac{\sin nx}{e^{2\pi x}-1}\,dx = \tfrac{1}{2}\left(\frac{1}{e^n-1}+\frac{1}{2}-\frac{1}{n}\right),$$

and

$$\int_0^\infty xe^{-x^2}\sin nx\,dx = \tfrac{1}{4}\sqrt{\pi}\,ne^{-\frac{1}{4}n^2},$$

we can prove that, if $\alpha\beta = \pi^2$, then

$$(13) \qquad \frac{1}{\sqrt[4]{\alpha}}\left\{1+2\alpha\int_0^\infty \frac{e^{-\alpha x}}{e^{2\pi\sqrt{x}}-1}\,dx\right\} = \frac{1}{\sqrt[4]{\beta}}\left\{1+2\beta\int_0^\infty \frac{e^{-\beta x}}{e^{2\pi\sqrt{x}}-1}\,dx\right\};$$

and so on.

5. Suppose now that a, b, and n are positive, and

$$(14) \qquad \int_0^\infty \phi(a,x)\begin{matrix}\cos\\ \sin\end{matrix} nx\,dx = \psi(a,n).$$

Then, if the conditions of Fourier's double integral theorem are satisfied, we have

$$(15) \qquad \int_0^\infty \psi(b,x)\begin{matrix}\cos\\ \sin\end{matrix} nx\,dx = \tfrac{1}{2}\pi\,\phi(b,n).$$

* Formulæ equivalent to (11) and (12) were given by Mr. Hardy, *Quarterly Journal*, vol. xxxv., p. 193.

Applying the formula (6) to (14) and (15), we obtain

$$(16)\quad \tfrac{1}{2}\pi \int_0^\infty \phi(a, x)\, \phi(b, nx)\, dx = \int_0^\infty \psi(b, x)\, \psi(a, nx)\, dx.$$

Thus, when $a = b$, we have the formula

$$\tfrac{1}{2}\pi \int_0^\infty \phi(x)\, \phi(nx)\, dx = \int_0^\infty \psi(x)\, \psi(nx)\, dx,$$

where
$$\psi(t) = \int_0^\infty \phi(x) \begin{matrix}\cos\\ \sin\end{matrix} tx\, dx;$$

and, in particular, if $n = 1$, then

$$\tfrac{1}{2}\pi \int_0^\infty \{\phi(x)\}^2\, dx = \int_0^\infty \{\psi(x)\}^2\, dx.$$

If
$$\phi(a, x) = \frac{1}{\{1 + x^2/a^2\}\{1 + x^2/(a+1)^2\}\dots} \qquad (a > 0),$$

then, by (1),

$$\psi(a, x) = \tfrac{1}{2}\sqrt{\pi}\, \frac{\Gamma(a + \frac{1}{2})}{\Gamma(a)}\, \operatorname{sech}^{2a} \tfrac{1}{2}x.$$

Hence, by (16),

$$\int_0^\infty \phi(a, x)\, \phi(b, x)\, dx = \frac{\Gamma(a + \frac{1}{2})\, \Gamma(b + \frac{1}{2})}{2\, \Gamma(a)\, \Gamma(b)} \int_0^\infty \operatorname{sech}^{2a+2b} \tfrac{1}{2}x\, dx;$$

and so

$$(17)\quad \int_0^\infty \frac{dx}{\{1 + x^2/a^2\}\{1 + x^2/(a+1)^2\}\dots\{1 + x^2/b^2\}\{1 + x^2/(b+1)^2\}\dots} = \tfrac{1}{2}\sqrt{\pi}\, \frac{\Gamma(a + \frac{1}{2})\, \Gamma(b + \frac{1}{2})\, \Gamma(a + b)}{\Gamma(a)\, \Gamma(b)\, \Gamma(a + b + \frac{1}{2})},$$

a and b being positive: or

$$(18)\quad \int_0^\infty |\Gamma(a + ix)\, \Gamma(b + ix)|^2\, dx = \tfrac{1}{2}\sqrt{\pi}\, \frac{\Gamma(a)\, \Gamma(a + \frac{1}{2})\, \Gamma(b)\, \Gamma(b + \frac{1}{2})\, \Gamma(a + b)}{\Gamma(a + b + \frac{1}{2})}.$$

As particular cases of the above result, we have, when $b = 1$,

$$\int_0^\infty \frac{x}{\sinh \pi x}\, \frac{dx}{\{1 + x^2/a^2\}\{1 + x^2/(a+1)^2\}\dots} = \frac{a}{2(1 + 2a)};$$

when $b = 2$,

$$\int_0^\infty \frac{x^3}{\sinh \pi x} \frac{dx}{\{1 + x^2/a^2\}\{1 + x^2/(a+1)^2\}\dots} = \frac{a^2}{2(1+2a)(3+2a)};$$

and so on. Since $\Pi\{1 + x^2/(a+n)^2\}$ can be expressed in finite terms by means of hyperbolic functions when $2a$ is an integer, we can deduce a large number of special formulæ from the preceding results.

6. Another curious formula is the following. If $0 < r < 1$, $n > 0$, and $0 < a < r^{n-1}$, then

$$(19) \quad \int_0^\infty \frac{(1+arx)(1+ar^2x)\dots}{(1+x)(1+rx)(1+r^2x)\dots} x^{n-1}dx = \frac{\pi}{\sin n\pi} \prod_{m=1}^{m=\infty} \frac{(1-r^{m-n})(1-ar^m)}{(1-r^m)(1-ar^{m-n})},$$

unless n is an integer or a is of the form r^p, where p is a positive integer.

If $a = r^p$, the formula reduces to

$$(20) \quad \int_0^\infty \frac{x^{n-1}dx}{(1+x)(1+rx)\dots(1+r^px)} = \frac{\pi}{\sin n\pi} \frac{(1-r^{1-n})(1-r^{2-n})\dots(1-r^{p-n})}{(1-r)(1-r^2)\dots(1-r^p)}.$$

If n is an integer, the value of the integral is in any case

$$-\frac{\log r}{1-a} \frac{(1-r)(1-r^2)\dots(1-r^{n-1})}{(r-a)(r^2-a)\dots(r^{n-1}-a)}.$$

My own proofs of the above results make use of a general formula, the truth of which depends on conditions which I have not yet investigated completely. A direct proof depending on Cauchy's theorem will be found in Mr. Hardy's note which follows this paper. The final formula used in Mr. Hardy's proof can be proved as follows. Let

$$f(t) = \prod_{m=0}^{m=\infty} \left(\frac{1-btx^m}{1-atx^m}\right) = A_0 + A_1t + A_2t^2 + \dots.$$

Then it is evident that

$$(1-at)f(t) = (1-bt)f(tx).$$

That is,

$$(1-at)(A_0+A_1t+A_2t^2+\ldots)=(1-bt)(A_0+A_1tx+A_2t^2x^2+\ldots).$$

Equating the coefficients of t^n, we obtain

$$A_n=A_{n-1}\frac{a-bx^{n-1}}{1-x^n};$$

and A_0 is evidently 1. Hence we have

$$(21)\quad f(t)=1+t\frac{a-b}{1-x}+t^2\frac{(a-b)(a-bx)}{(1-x)(1-x^2)}+\ldots.$$

7. As a particular case of (19), we have, when $a=0$,

$$(22)\quad \int_0^\infty \frac{x^{n-1}dx}{(1+x)(1+rx)(1+r^2x)\ldots}=\frac{\pi}{\sin n\pi}\frac{1-r^{1-n}}{1-r}\frac{1-r^{2-n}}{1-r^2}\ldots.$$

When n is an integer, the value of the integral reduces to

$$-r^{-\frac{1}{2}n(n-1)}(1-r)(1-r^2)\ldots(1-r^{n-1})\log r.$$

When we put $n=\frac{1}{2}$ in (19), we have

$$(23)\quad \int_0^\infty \frac{1}{1+x^3}\frac{1+ar^2x^2}{1+r^2x^2}\frac{1+ar^4x^2}{1+r^4x^2}\ldots dx$$
$$=\tfrac{1}{2}\pi\frac{1-ar^2}{1-r^2}\frac{1-ar^4}{1-r^4}\ldots\frac{1-r}{1-ar}\frac{1-r^3}{1-ar^3}\ldots.$$

If, in particular, $n=\frac{1}{2}$ in (22), or $a=0$ in (23), then

$$(24)\quad \int_0^\infty \frac{dx}{(1+x^3)(1+r^2x^2)(1+r^4x^3)\ldots}$$
$$=\tfrac{1}{2}\pi\frac{1-r}{1-r^2}\frac{1-r^3}{1-r^4}\frac{1-r^5}{1-r^6}\ldots$$
$$=\frac{\pi}{2(1+r+r^3+r^6+r^{10}+\ldots)},$$

the n^{th} term in the denominator being $r^{\frac{1}{2}n(n-1)}$. Thus, for example, when $r=e^{-5\pi}$, we have

$$\int_0^\infty \frac{dx}{(1+x^2)(1+e^{-10\pi}x^2)(1+e^{-20\pi}x^2)\ldots}$$
$$=\frac{\pi}{2(1+e^{-5\pi}+e^{-15\pi}+e^{-30\pi}+\ldots)}$$
$$=\pi^{\frac{3}{4}}\Gamma(\tfrac{3}{4})\sqrt{5}\sqrt[8]{2}\,\tfrac{1}{2}(1+\sqrt[4]{5})\{\tfrac{1}{2}(1+\sqrt{5})\}^{\frac{1}{2}}e^{-5\pi/8}.$$

Similarly

$$\int_0^\infty \frac{dx}{\sqrt{(1+x^2)}\sqrt{(1+e^{-20\pi x^2})}\sqrt{(1+e^{-40\pi x^2})}\dots}$$
$$= \pi^{\frac{3}{4}}\Gamma\left(\tfrac{3}{4}\right)\sqrt{5}\sqrt[4]{2}\,\tfrac{1}{2}(1+\sqrt[4]{5})^2\left\{\tfrac{1}{2}(1+\sqrt{5})\right\}^{\frac{5}{2}}e^{-5\pi/4};$$

and

$$\int_0^\infty \frac{dx}{\sqrt{(1+x^2)}\sqrt{(1+\cdot001x^2)}\sqrt{(1+\cdot00001x^2)}\dots}$$
$$= \tfrac{1}{2}\pi\,\frac{10}{11}\,\frac{1110}{1111}\,\frac{111110}{111111}\cdots$$
$$= \frac{\pi}{2.202\,002\,000\,200\,002\,000\,002\dots}.$$

PROOF OF A FORMULA OF Mr. RAMANUJAN.

By *G. H. Hardy.*

1. The formula in question is

$$(1)\quad \int_0^\infty \frac{(1+abx)(1+ab^2x)\dots}{(1+x)(1+bx)(1+b^2x)\dots}x^{n-1}dx$$
$$= \frac{\pi}{\sin n\pi}\prod_{m=1}^{\infty}\frac{(1-b^{m-n})(1-ab^m)}{(1-ab^{m-n})(1-b^m)},$$

where n, b, and a are positive, b and a are less than 1, and $a<b^{n-1}$. The last condition is, of course, included in the others if $n\leq 1$. I shall suppose that n is not an integer and a is not a power of b. The formulæ relating to these exceptional cases will be found in Mr. Ramanujan's paper, and are easily deduced from (1).

Consider the function

$$F(z)=(1+z)(1+bz)(1+b^2z)\dots.$$

It is an integral function of zero order, having zeroes at the points $z=-b^{-p}$ ($p=0, 1, 2, \dots$).

I shall apply Cauchy's Theorem to the integral

$$\int_C \frac{F(abz)}{F(z)}(-z)^{n-1}dz,$$

where $(-z)^{n-1}=e^{(n-1)\log(-z)}$, $\log(-z)$ having its principal

value, and the contour C is formed by (i) the segment (ρ, R) of the real axis, (ii) the circle $|z|=R$, described positively, (iii) the segment (R, ρ) of the real axis, (iv) the circle $|z|=\rho$, described negatively. I suppose that R is of the form $b^{-p-1/2}$, where p is an integer.

The sole difficulty in the proof lies in showing that the integral round the large circle tends to zero as $R\to\infty$. If we assume this for the moment, we obtain, when $R\to\infty$ and $\rho\to 0$,

$$(2) \qquad 2i\sin n\pi\int_0^\infty \frac{F(abx)}{F(x)}\,x^{n-1}dx = 2\pi i\,\Sigma E_p,$$

where E_p is the residue at the pole $z=-b^{-p}$.

2. It has been proved by Mellin* that, if we exclude from the plane of $z=re^{i\theta}$ the sector $\pi-\delta<\theta<\pi+\delta$, then

$$F(z)=\exp\left[\frac{\{\log z+\frac{1}{2}\log(1/b)\}^2}{2\log(1/b)}\right]\phi(z),$$

where
$$H<|\phi(z)|<K,$$

H and K being positive constants.

Further, I have shown† that, in the region of the plane excluded from the scope of Mellin's formula,

$$F(z)=\exp\left[\frac{\{\log(-z)+\frac{1}{2}\log(1/b)\}^2}{2\log(1/b)}\right]\sin\left\{\frac{\pi\log(-z)}{\log(1/b)}\right\}\phi(z),$$

where $\phi(z)$ satisfies the same conditions as before. It follows that, on the circle $|z|=R=b^{-p-1/2}$, we have

$$|F(z)|>A\sqrt{R}\exp\left\{\frac{(\log R)^2}{2\log(1/b)}\right\},$$

where A is a positive constant. But we also have

$$|F(abz)|<B\sqrt{R}\exp\left\{\frac{(\log abR)^2}{2\log(1/b)}\right\},$$

and so

$$\left|\frac{F(abz)}{F(z)}\right|<C\exp\left\{\frac{(\log abR)^2-(\log R)^2}{2\log(1/b)}\right\}<DR^{-\nu},$$

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* *Acta Societatis Fennicae*, vol. xxix, No. 4, p. 15.

† *Quarterly Journal*, vol. xxxvii, p. 172. A factor $\frac{1}{2}$ is omitted there in the index of the exponential.

where B, C, D are constants and

$$\nu = 1 + \frac{\log(1/a)}{\log(1/b)}.$$

But $a < b^{n-1}$, and so

$$\frac{\log(1/a)}{\log(1/b)} > n-1,$$

$$n-1-\nu < -1.$$

From the last inequality it follows at once that the integral along the large circle tends to zero.

3. Hence, from (2),

$$\int_0^\infty \frac{F(abx)}{F(x)}\, x^{n-1}\, dx = \frac{\pi}{\sin n\pi}\, \Sigma E_p.$$

But

$$E_p = b^{-np}\, \frac{(1-ab^{1-p})(1-ab^{2-p})(1-ab^{3-p})\ldots}{(1-b^{-p})(1-b^{1-p})\ldots(1-b^{-1})(1-b)(1-b^2)\ldots}.$$

Hence, after a little reduction, we obtain, as the value of the integral,

$$\frac{\pi}{\sin n\pi}\, \frac{(1-ab)(1-ab^2)\ldots}{(1-b)(1-b^2)\ldots}\, \sum_0^\infty \frac{(ab-b)(ab-b^2)\ldots(ab-b^p)}{(1-b)(1-b^2)\ldots(1-b^p)}\, b^{-np},$$

which is equal to

$$\frac{\pi}{\sin n\pi}\, \frac{(1-ab)(1-ab^2)\ldots}{(1-b)(1-b^2)\ldots}\, \frac{(1-b^{1-n})(1-b^{2-n})\ldots}{(1-ab^{1-n})(1-ab^{2-n})\ldots},$$

in virtue of Mr. Ramanujan's formula (21).*

4. I have succeeded in evaluating another curious integral involving the function $F(z)$, viz.,

$$(3) \qquad I = \int_0^\infty \exp\left\{-\frac{(\log x)^2}{2\log(1/b)}\right\} F(ab^\nu x)\, x^{\frac{1}{2}-\nu}\, dx.$$

The condition that this integral should be convergent is easily found to be

$$\nu + \frac{\log(1/a)}{\log(1/b)} > 0.$$

* Writing b for x, ab for a, and b^{-n} for t.

If we put $x = by$, we obtain

$$I = \frac{1}{ab^{\nu}} \int_0^{\infty} \exp\left\{-\frac{(\log y)^2}{2\log(1/b)}\right\} F(ab^3 y)\, \frac{ab^2 y}{1+ab^2 y}\, y^{\frac{1}{2}-\nu}\, dy,$$

from which we deduce that

$$I = \frac{I_1}{1-ab^{\nu}},$$

where

$$I_1 = \int_0^{\infty} \exp\left\{-\frac{(\log x)^2}{2\log(1/b)}\right\} F(ab^3 x)\, x^{\frac{1}{2}-\nu}\, dx.$$

Repeating this argument, we find

$$I = \frac{I_p}{(1-ab^{\nu})(1-ab^{\nu+1})\ldots(1-ab^{\nu+p-1})},$$

where

$$I_p = \int_0^{\infty} \exp\left\{-\frac{(\log x)^2}{2\log(1/b)}\right\} F(ab^{p+2} x)\, x^{\frac{1}{2}-\nu}\, dx.$$

But it is easy to prove that, as $p \to \infty$,

$$I_p \to \int_0^{\infty} \exp\left\{-\frac{(\log x)^2}{2\log(1/b)}\right\} x^{\frac{1}{2}-\nu}\, dx = \sqrt{\{2\pi \log(1/b)\}}\, b^{-\frac{1}{2}(\frac{3}{2}-\nu)^2}.$$

Hence

$$(5) \qquad I = \frac{\sqrt{\{2\pi \log(1/b)\}}\, b^{-\frac{1}{2}(\frac{3}{2}-\nu)^2}}{(1-ab^{\nu})(1-ab^{\nu+1})(1-ab^{\nu+2})\ldots}.$$

ON THE RATIONAL DETERMINATION OF A PLANE QUARTIC FROM SEVEN GIVEN BITANGENTS.

By *W. Burnside.*

I DO not think it has been noticed that Hesse's* method of treating the plane quartic can be used to simplify materially Aronhold's† rational determination of the quartic from seven bitangents.

Hesse shows that, when one root of an equation of the 36th degree has been determined, a quartic curve can be expressed in the form

$$F(\lambda, \mu, \nu) = 0,$$

* *Crelle's Journal*, vol. xlix.
† *Berliner Monats berichte* (1864).

where $F(\lambda, \mu, \nu)$ is the discriminant of a doubly infinite system of quadric surfaces

$$\lambda f + \mu g + \nu h = 0.$$

Further, he shows that if the coefficient of k in

$$f(x_1 + kx_2,\ y_1 + ky_2,\ z_1 + kz_2,\ u_1 + ku_2)$$

is f_{12}, and if (x_i, y_i, z_i, u_i) $(i = 1, 2, \ldots, 8)$ are the eight common points of the system of quadric surfaces, then

$$\lambda f_{ij} + \mu g_{ij} + \nu h_{ij} = 0 \quad (ij = 1, 2, \ldots, 8)$$

are the twenty-eight bitangents of

$$F(\lambda, \mu, \nu) = 0.$$

Seven arbitrary lines in a plane can, by a suitable projective transformation, be brought to form

$$\lambda = 0, \quad \mu = 0, \quad \nu = 0, \quad \lambda + \mu + \nu = 0,$$

$$a_1\lambda + a_2\mu + a_3\nu = 0, \quad b_1\lambda + b_2\mu + b_3\nu = 0, \quad c_1\lambda + c_2\mu + c_3\nu = 0.$$

Having effected this transformation, determine a_4, b_4, c_4 to satisfy the linear equations

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a_4a_1 & a_2a_3 & a_3a_1 & a_1a_2 \\ b_4b_1 & b_2b_3 & b_3b_1 & b_1b_2 \\ c_4c_1 & c_2c_3 & c_3c_1 & c_1c_2 \end{vmatrix} = 0, \qquad \begin{vmatrix} 1 & 1 & 1 & 1 \\ a_4a_2 & a_2a_3 & a_3a_1 & a_1a_2 \\ b_4b_2 & b_2b_3 & b_3b_1 & b_1b_2 \\ c_4c_2 & c_2c_3 & c_3c_1 & c_1c_2 \end{vmatrix} = 0,$$

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a_4a_2 & a_2a_3 & a_3a_1 & a_1a_2 \\ b_4b_3 & b_2b_3 & b_3b_1 & b_1b_2 \\ c_4c_3 & c_2c_3 & c_3c_1 & c_1c_2 \end{vmatrix} = 0.$$

Then the four points

$$(1, 1, 1, 1), \quad (a_1, a_2, a_3, a_4), \quad (b_1, b_2, b_3, b_4), \quad (c_1, c_2, c_3, c_4)$$

lie on the three quadrics

$$f \equiv x_4x_1 + A\,x_2x_3 + B\,x_3x_1 + C\,x_1x_2 = 0,$$
$$g \equiv x_4x_2 + A'\,x_2x_3 + B'\,x_3x_1 + C'\,x_1x_2 = 0,$$
$$h \equiv x_4x_3 + A''x_2x_3 + B''x_3x_1 + C''x_1x_2 = 0;$$

where A, B, C are determined rationally by

$$a_4a_1 + Aa_2a_3 + Ba_3a_1 + Ca_1a_2 = 0,$$

$$b_4b_1 + Ab_2b_3 + Bb_3b_1 + Cb_1b_2 = 0,$$

$$c_4c_1 + Ac_2c_3 + Bc_3c_1 + Cc_1c_2 = 0;$$

while A', B', C' and A'', B'', C'' are determined by similar sets of equations.

The quadrics $f=0$, $g=0$, $h=0$ also pass through the corners of the tetrahedron of reference.

Now the discriminant of $\lambda f + \mu g + \nu h = 0$ is

$$F(\lambda, \mu, \nu) \equiv \lambda^2(A\lambda + A'\mu + A''\nu)^2 + \mu^2(B\lambda + B'\mu + B''\nu)^2$$
$$+ \nu^2(C\lambda + C'\mu + C''\nu)^2 - 2\mu\nu(B\lambda + B'\mu + B''\nu)$$
$$- 2\nu\lambda(C\lambda + C'\mu + C''\nu)(A\lambda + A'\mu + A''\nu)$$
$$- 2\lambda\mu(A\lambda + A'\mu + A''\nu)(B\lambda + B'\mu + B''\nu).$$

Since the coordinates of the intersections of the three quadrics are rational in the coefficients of the equations of the seven lines, the equations of the bitangents of

$$F(\lambda, \mu, \nu) = 0$$

are rational in the same quantities.

If the eight points of intersection of the quadrics

$$(0, 0, 0, 1), \quad (1, 0, 0, 0), \quad (0, 1, 0, 0), \quad (0, 0, 1, 0),$$

$$(1, 1, 1, 1), \quad (a_1, a_2, a_3, a_4), \quad (b_1, b_2, b_3, b_4), \quad (c_1, c_2, c_3, c_4)$$

are denoted by 1, 2, 3, 4, 5, 6, 7, 8, it is found that the bitangents corresponding to the chords 12, 13, 14, 15, 16, 17, 18 are respectively the seven lines above the order written. These lines therefore form an Aronhold system for the quartic

$$F(\lambda, \mu, \nu) = 0,$$

and the remaining twenty-one bitangents are rationally determinable.

AN ELEMENTARY PROOF OF STAUDT'S THEOREM ON BERNOULLI'S NUMBERS.

By *S. T. Shovelton.*

FROM the identity

$$\frac{n!}{\{1-nx\}\{1-(n-1)x\}\ldots\{1-x\}}$$

$$= \left[\frac{n^n}{1-nx} - {}^nC_1\frac{(n-1)^n}{1-(n-1)x} + {}^nC_2\frac{(n-2)^n}{1-(n-2)x} - \ldots\right]$$

$$= \left[\{n^n - {}^nC_1(n-1)^n + \ldots\} + x\{n^{n+1} - {}^nC_1(n-1)^{n+1} + \ldots\} + \ldots\right]$$

(for $|x| < n^{-1}$)

$$= \Delta^n 0^n + x\Delta^n 0^{n+1} + x^2\Delta^n 0^{n+2} + \ldots$$

it follows that $\dfrac{\Delta^n 0^{n+r}}{n!}$ is an integer, and therefore that $\dfrac{\Delta^\kappa}{\kappa+1}0^{2n}$ is an integer if $(\kappa+1)$ is not prime.

Now

$$(-1)^{n-1}B_n = \left(1 - \frac{\Delta}{2} + \frac{\Delta^2}{3} - \ldots + \frac{\Delta^{2n}}{2n+1}\right)0^{2n}.$$

Therefore, since $\dfrac{\Delta^\kappa}{\kappa+1}0^{2n}$ is an integer if $(\kappa+1)$ is not prime, it follows that

$$(-1)^{n-1}B_n = -\tfrac{1}{2} + \Sigma\frac{\Delta^{2\kappa}}{2\kappa+1}0^{2n} + \text{an integer},$$

where the summation is taken for all values of κ from 1 to n, such that $2\kappa+1$ is prime.

Let $n = \mu\kappa + \alpha$, where $2\kappa+1$ is a prime number $\not> n$, and let α be $< \kappa$. Then

$$\Delta^{2\kappa}0^{2n} = (2\kappa)^{2n} - {}^{2\kappa}C_1(2\kappa-1)^{2n} \ldots + 0^{2n}$$

$$= (2\kappa)^{2\mu\kappa+2\alpha} - {}^{2\kappa}C_1(2\kappa-1)^{2\mu\kappa+2\alpha} + \ldots.$$

Now

$$(2\kappa-r)^{2\mu\kappa} = \{(2\kappa-r)^\mu\}^{2\kappa} = 1 + M(2\kappa+1)$$

by Fermat's theorem. Therefore

$$\Delta^{2\kappa}0^{2n} = M(2\kappa+1) + (2\kappa)^{2\alpha} - {}^{2\kappa}C_1(2\kappa-1)^{2\alpha} + \ldots$$

$$= M(2\kappa+1) + \Delta^{2\kappa}0^{2\alpha} \quad (\text{unless } \alpha \text{ is zero})$$

$$= M(2\kappa+1),$$

since κ is greater than α. But if α is zero, *i.e.*, $n=\mu\kappa$,

$$\Delta^{2\kappa}0^{2n}=M(2\kappa+1)+1-{}^{2\kappa}C_1+{}^{2\kappa}C_2\ldots-{}^{2\kappa}C_{2\kappa-1}.$$
$$=M(2\kappa+1)-1.$$

Therefore

$$(-1)^{n-1}B_n=I-\tfrac{1}{2}-\Sigma\frac{1}{2\kappa+1},$$

where I is an integer, and the summation is taken for all values of κ, including unity, which are factors of n and are such that $2\kappa+1$ is prime.

NOTE ON THE DIFFERENTIAL EQUATION FOR MINIMUM SURFACES.

By *A. M. Grundy.*

THE following note on M. Darboux's *Théorie des Surfaces* may be of interest.

On pp. 274 and 275 of vol. 1, or bk. iii., ch. 1, § 179, a differential equation for minimum surfaces is given in the form

$$\left[\left(\frac{\partial x}{\partial a}\right)^2+\left(\frac{\partial y}{\partial a}\right)^2+\left(\frac{\partial z}{\partial a}\right)^2\right]\left(X\frac{\partial^2 x}{\partial b^2}+Y\frac{\partial^2 y}{\partial b^2}+Z\frac{\partial^2 z}{\partial b^2}\right)$$
$$-2\left(\frac{\partial x}{\partial a}\frac{\partial x}{\partial b}+\frac{\partial y}{\partial a}\frac{\partial y}{\partial b}+\frac{\partial z}{\partial a}\frac{\partial z}{\partial b}\right)\left(X\frac{\partial^2 x}{\partial a\,\partial b}+Y\frac{\partial^2 y}{\partial a\,\partial b}+Z\frac{\partial^2 z}{\partial a\,\partial b}\right)$$
$$+\left[\left(\frac{\partial x}{\partial b}\right)^2+\left(\frac{\partial y}{\partial b}\right)^2+\left(\frac{\partial z}{\partial b}\right)^2\right]\left(X\frac{\partial^2 x}{\partial a^2}+Y\frac{\partial^2 y}{\partial a^2}+Z\frac{\partial^2 z}{\partial a^2}\right)=0\ldots(6),$$

in which a, b are two independent parameters, x, y, z are functions of a and b, and X, Y, Z the direction cosines of the normal to the surface.

Calling $\partial x/\partial a$, x_1, etc., and

$$x_1^2+y_1^2+z_1^2=e,\quad x_1x_2+y_1y_2+z_1z_2=f,\quad x_2^2+y_2^2+z_2^2=g,$$

Monge remarked that (6) is satisfied by $e=0$, $g=0$, and

$$x_{12}=0,\ y_{12}=0,\ z_{12}=0;$$

while M. Darboux observes that putting $e=0$, $g=0$ leads to

$$Xx_{12}+Yy_{12}+Zz_{12}=0.$$

In point of fact the three equations

$$e=0,\ g=0,\ Xx_{12}+Yy_{12}+Zz_{12}=0$$

involve, as a necessary consequence,

$$x_{12}=0,\ y_{12}=0,\ z_{12}=0.$$

For, using the identities

$$Xx_1+Yy_1+Zz_1=0,$$
$$Xx_2+Yy_2+Zz_2=0,$$

we have $$\frac{X}{y_1z_2-y_2z_1}=\frac{Y}{z_1x_2-z_2x_1}=\frac{Z}{x_1y_2-x_2y_1}=\frac{1}{h},$$

where $h^2=eg-f^2$, and

$$\begin{vmatrix} X, & Y, & Z \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{vmatrix}=h.$$

So $$Xx_{12}+Yy_{12}+Zz_{12}=0$$

involves

$$(y_1z_2-y_2z_1)\,x_{12}+(z_1x_2-z_2x_1)\,y_{12}+(x_1y_2-x_2y_1)\,z_{12}=0;$$

but, since $e=0$, $g=0$,

$$x_1x_{12}+y_1y_{12}+z_1z_{12}=\tfrac{1}{2}\frac{\partial e}{\partial b}=0;$$

$$x_2x_{12}+y_2y_{12}+z_2z_{12}=\tfrac{1}{2}\frac{\partial g}{\partial a}=0;$$

and the determinant

$$\begin{vmatrix} y_1z_2-y_2z_1, & z_1x_2-z_2x_1, & x_1y_2-x_2y_1 \\ x_1\ , & y_1\ , & z_1 \\ x_2\ , & y_2\ , & z_2 \end{vmatrix}$$

$$=h\begin{vmatrix} X, & Y, & Z \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{vmatrix}=h^2=eg-f^2=-(x_1x_2+y_1y_2+z_1z_2)^2$$

in this case, and so cannot be zero; and therefore

$$x_{12}=0,\quad y_{12}=0,\quad z_{12}=0.$$

THE ORTHOCENTRIC $(n+2)$-ANGLE.

By *T. C. Lewis, M.A.*

1. The simplest name for the figure in n-dimensional space analogous to the orthocentric tetrahedron in three dimensions, completed by regarding the orthocentre itself as one of the vertices of the figure, appears to be the orthocentric $(n+2)$-angle, which has been employed by Schoute.

2. Hyper spheres may be described with the vertices as centres so that every pair cut one another orthogonally. If their radii at the successive vertices are $\rho_1, \rho_2, \ldots, \rho_{n+2}$, and if the Cartesian coordinates of the k^{th} vertex are a_k, b_k, c_k, &c., the following geometrical properties have been established, and are given here for the convenience of those who are not familiar with the Darboux coordinates which will be used in this paper:

$$\Sigma \frac{1}{\rho_k^2} = 0,$$

$$\Sigma \frac{a_k}{\rho_k^2} = \Sigma \frac{b_k}{\rho_k^2} = \&c. = 0,$$

$$\Sigma \frac{a_k^2}{\rho_k^2} = 1,$$

$$\Sigma \frac{a_k b_k}{\rho_k^2} = 0.$$

Therefore

$$\Sigma \frac{a_k^2 + b_k^2 + \ldots - \rho_k^2}{\rho_k^2} = n - (n+2) = -2,$$

and

$$\Sigma \frac{a_k(a_k^2 + b_k^2 + \ldots - \rho_k^2)}{\rho_k^2}$$

$$= \Sigma \frac{(a_k - a_{n+2})\{(a_k - a_{n+2})^2 + (b_k - b_{n+2})^2 + \ldots - \rho_k^2\}}{\rho_k^2}$$

$$= \Sigma \frac{(a_k - a_{n+2})\rho_{n+2}^2}{\rho_k^2},$$

$$= 0;$$

also

$$\Sigma \frac{(a_k^2+b_k^2+\ldots-\rho_k^2)^2}{\rho_k^2} = \Sigma \frac{\{(a_k-a_{n+2})^2+(b_k-b_{n+2})^2+\ldots-\rho_k^2\}^2}{\rho_k^2}$$
$$= \Sigma \frac{\rho_{n+2}^4}{\rho_k^2}$$
$$= 0.$$

Hence for any point

$$\Sigma \frac{(x-a_k)^2+(y-b_k)^2+\ldots-\rho_k^2}{\rho_k^2} = -2 \ldots\ldots (1)$$

and

$$\Sigma \frac{\{(x-a_k)^2+(y-b_k)^2+\ldots-\rho_k^2\}^2}{\rho_k^2} = 0 \ldots\ldots (2).$$

[If the point is at a perpendicular distance t outside the n-space considered, the value of the latter expression becomes $-4t^2$ instead of zero, $(x-a_k)^2+(y-b_k)^2+\ldots$ being the square of the distance from the point $(a_k, b_k, \ldots)$.]

After Gaston Darboux, take as the $n+2$ coordinates of a point in n dimensions $x_1, x_2, \ldots, x_{n+2}$, where

$$x_k = \frac{(x-a_k)^2+(y-b_k)^2+\ldots-\rho_k^2}{\rho_k} = \frac{S_k}{\rho_k}.$$

Then there are two relations between the coordinates, viz.,

$$\Sigma \frac{x_k}{\rho_k} = -2$$

and

$$\Sigma x_k^2 = 0.$$

3. The equation to any n-sphere S may be written in the form

$$\frac{S}{\rho} \equiv \alpha_1 \frac{S_1}{\rho_1} + \alpha_2 \frac{S_2}{\rho_2} + \ldots + \alpha_{n+2} \frac{S_{n+2}}{\rho_{n+2}} = 0 \ldots\ldots (3),$$

where S is the square of the tangent from any point and ρ the radius of the sphere, appropriate values being given to the coefficients α.

To determine these coefficients, K being the centre of S, and A of S_1, and c_k the cosine of the angle at which S and S_k intersect,

$$AK^2 = \rho^2 + \rho_1^2 - 2c_1\rho\rho_1.$$

Therefore

$$\frac{AK^2-\rho_1^2}{\rho_1^2} = \frac{\rho^2}{\rho_1^2} - 2\rho\frac{c_1}{\rho_1}.$$

Therefore

$$\Sigma \frac{AK^2-\rho_k^2}{\rho_k^2} = -2\rho \Sigma \frac{c_k}{\rho_k}.$$

Therefore $$\frac{1}{\rho} = \Sigma \frac{c_k}{\rho_k}.$$

If the identity (3) be applied to the $n+2$ vertices of the figure, in succession, $n+2$ equations are obtained similar to

$$-\frac{2\alpha_1}{\rho_1} + \Sigma \frac{\alpha_k}{\rho_k} = \frac{1}{\rho} - \frac{2c_1}{\rho_1}.$$

Then, by addition,

$$n\Sigma \frac{\alpha_k}{\rho_k} = \frac{n+2}{\rho} - 2\Sigma \frac{c_k}{\rho_k}$$

$$= \frac{n}{\rho},$$

therefore $$\Sigma \frac{\alpha_k}{\rho_k} = \frac{1}{\rho} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Substituting this in each of the $n+2$ equations, we obtain, in general,

$$\alpha_k = c_k,$$

i.e., α_k is the cosine of the angle of intersection of S and S_k.

4. Apply the same identity to the centre of S. Then

$$-\rho = \alpha_1 \frac{\rho(\rho - 2\alpha_1\rho_1)}{\rho_1} + \&c.$$

$$= \rho^2 \Sigma \frac{\alpha_k}{\rho_k} - 2\rho\Sigma\alpha_k^2.$$

Therefore $$\Sigma\alpha_k^2 = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

5. If there are two n-spheres,

$$\frac{S}{\rho} \equiv \Sigma\left(\alpha_k \frac{S_k}{\rho_k}\right) = 0,$$

and $$\frac{S'}{\rho'} \equiv \Sigma\left(\alpha_k' \frac{S_k}{\rho_k}\right) = 0,$$

apply the first identity to the centre of the second of these n-spheres, then, if c be the cosine of the angle of intersection of S and S',

$$\frac{\rho'(\rho' - 2c\rho)}{\rho} = \Sigma\alpha_k \frac{\rho'(\rho' - 2\alpha_k'\rho_k)}{\rho_k},$$

whence $$c = \Sigma\alpha_k\alpha_k' \quad \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

6. The equation of any n-sphere, in terms of the coordinates which have been adopted, is

$$\Sigma \alpha_k x_k = 0,$$

and the radius has been determined in terms of the coefficients and the constants of the $(n+2)$-angle.

The coordinates of the centre are given by

$$x_k' = \frac{\rho(\rho - 2\alpha_k \rho_k)}{\rho_k} \quad \ldots\ldots\ldots\ldots\ldots\ldots (7).$$

Thus the equation to an n-sphere of radius ρ and centre at a given point $(x_1', x_2', \ldots)$ may at once be written down, viz.,

$$\Sigma \left(x_k' - \frac{\rho^2}{\rho_k} \right) x_k = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

It may also be noticed that at any point on S,

$$\begin{aligned} \Sigma (x_k - x_k')^2 &= -2\Sigma x_k x_k' \\ &= -2\rho \Sigma \frac{S_k}{\rho_k^2} (\rho - 2\alpha_k \rho_k) \\ &= 4\rho^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9). \end{aligned}$$

. If the successive vertices be A, B, ..., N, P, then at A, viz., at the orthocentre of the figure whose vertices are the remaining $n+1$ points,

$$\begin{aligned} \rho_2 x_2 = \rho_3 x_3 = \ldots = \rho_{n+2} x_{n+2} &= \rho_1^2 \quad \ldots\ldots\ldots (10) \\ &= -\rho_1 x_1. \end{aligned}$$

So, for the line AB,

$$\rho_3 x_3 = \rho_4 x_4 = \ldots = \rho_{n+2} x_{n+2};$$

and generally for the r-face, determined by the first $r+1$ of the $n+2$ given vertices, the following $n-r$ equations hold good, viz.,

$$\rho_{r+2} x_{r+2} = \rho_{r+3} x_{r+3} = \ldots = \rho_{n+2} x_{n+2}.$$

If $r = n-1$ there will remain but one equation:

$$\rho_{n+1} x_{n+1} = \rho_{n+2} x_{n+2}.$$

Thus the equations of the orthocentre of the first $r+1$ points are

$$\rho_1 x_1 = \rho_2 x_2 = \ldots = \rho_{r+1} x_{r+1}$$

together with

$$\rho_{r+2} x_{r+2} = \rho_{r+3} x_{r+3} = \ldots = \rho_{n+2} x_{n+2},$$

where

$$\rho_1 x_1 = -\rho_{r+2} x_{r+2}.$$

It is thus clear that the orthocentre of any number $r+1$ of the given vertices coincides with the orthocentre of the remaining $n-r+1$ vertices. Therefore at the orthocentre of the r-face here considered

$$\sum_1^{n+1} \rho_k x_k = (r+1)\,\rho_1 x_1 + (n-r)\,\rho_{n+2} x_{n+2} = (n-2r-1)\,\rho_{n+2} x_{n+2} \quad \ldots\ldots(11).$$

But this, being homogeneous of the first degree, is the equation to an n-sphere; and from its symmetry it follows that the orthocentres of all the $\dfrac{(n+1)!}{(r+1)!\,(n-r)!}$ r-faces of the figure, of which $A, \ldots, N$ are vertices and P the orthocentre, lie on this same n-sphere, whose radius ρ_r is given by

$$\frac{1}{\rho_r} = \frac{2\,(r+1)}{\sqrt{\{\sum_1^{n+1} \rho_k^{\,2} + (n-2r-1)^2 \rho_{n+2}^{\,2}\}}}$$

or
$$4\,(r+1)^2 \rho_r^{\,2} = \sum_1^{n+1} \rho_k^{\,2} + (n-2r-1)^2 \rho_{n+2}^{\,2} \ldots\ldots(12).$$

If $r=0$, the circumradius R is determined by

$$4R^2 = \sum_1^{n+1} \rho_k^{\,2} + (n-1)^2 \rho_{n+2}^{\,2},$$

whence
$$(r+1)^2 \rho_r^{\,2} = R^2 - r\,(n-r-1)\,\rho_{n+2}^{\,2}.$$

8. If $x_1', \ldots, x_{n+2}'$ are the coordinates of O_r, the centre of the n-sphere, then, as in (7),

$$\rho_k x_k' = \rho_r(\rho_r - 2\alpha_k \rho_k).$$

Therefore
$$\rho_k x_k' = \rho_r^{\,2} - \rho_k^{\,2}/(r+1),$$

where k has any integral value from 1 to $n+1$ inclusive; therefore $\dfrac{\rho_h x_h' - \rho_k \rho_k'}{\rho_h^{\,2} - \rho_k^{\,2}}$ is independent of h and k; therefore O_r lies on the line given by the following $n-1$ equations:

$$\frac{\rho_1 x_1 - \rho_2 x_2}{\rho_1^{\,2} - \rho_2^{\,2}} = \frac{\rho_2 x_2 - \rho_3 x_3}{\rho_2^{\,2} - \rho_3^{\,2}} = \ldots = \frac{\rho_n x_n - \rho_{n+1} x_{n+1}}{\rho_n^{\,2} - \rho_{n+1}^{\,2}} \ldots(13),$$

which are independent of r; therefore the centres of all the n-spheres corresponding to different values of r lie on this

straight line, which must therefore pass through the circumcentre O, and it obviously passes through P, the orthocentre of the figure. Moreover,

$$\rho_{n+2}x'_{n+2} = \frac{\sum_1^{n+1} \rho_k^2 + (n-2r-1)^2\rho^2_{n+2} + 4(r+1)(n-2r-1)\rho^2_{n+2}}{4(r+1)^2}.$$

Therefore

$$PO_r^2 = \rho_{n+2}x'_{n+2} + \rho^2_{n+2} = \frac{\sum_1^{n+1} \rho_k^2 + (n+1)^2\rho^2_{n+2}}{4(r+1)^2}.$$

Therefore $$4PO^2 = 4l^2 = \sum_1^{n+1} \rho_k^2 + (n+1)^2\rho^2_{n+2}.$$

Therefore $$PO_r = \frac{l}{r+1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (14).$$

Thus O_r is on PO at $\frac{1}{r+1}$ of its length from the orthocentre P.

The line joining the circumcentre and orthocentre of the r-face containing the first $r+1$ points is given by the $n-1$ equations

$$\frac{\rho_1 x_1 - \rho_2 x_2}{\rho_1^2 - \rho_2^2} = \frac{\rho_2 x_2 - \rho_3 x_3}{\rho_2^2 - \rho_3^2} = \ldots = \frac{\rho_r x_r - \rho_{r+1} x_{r+1}}{\rho_r^2 - \rho^2_{r+1}}$$

and $$\rho_{r+2}x_{r+2} = \rho_{r+3}x_{r+3} = \ldots = \rho_{n+2}x_{n+2}.$$

This line is therefore the projection of OP on the r-face.

9. It is known that the centroid G of the $n+1$ points $A, B, \ldots, N$ lies on PO at a distance $\frac{2l}{n+1}$ from P. Its coordinates are therefore obtained by writing $n+1$ for $2(r+1)$ in the general expressions for those of the centre of the n-sphere that passes through the orthocentres of all r-faces.

Thus, at G, $$x_k = \frac{\sum_1^{n+1} \rho_k^2}{(n+1)^2\rho_k} - \frac{2\rho_k}{n+1}$$

for values of k from 1 to $n+1$, and

$$x_{n+2} = \frac{\sum_1^{n+1} \rho_k^2}{(n+1)^2\rho_{n+2}}.$$

And for the centroid of the first $r+1$ vertices, *i.e.*, of the r-face containing them,

$$x_k = \frac{\sum_1^{r+1} \rho_k^2}{(r+1)^2 \rho_k} - \frac{2\rho_k}{r+1}$$

for values of k from 1 to $r+1$, and

$$\rho_{r+2} x_{r+2} = \rho_{r+3} x_{r+3} = \ldots = \frac{\sum_1^{r+1} \rho_k^2}{(r+1)^2},$$

therefore

$$\sum_1^{n+1} \rho_k x_k = \frac{\sum_1^{r+1} \rho_k^2}{r+1} \left(\frac{n-r}{r+1} - 1 \right)$$

$$= (n-2r-1)\, \rho_{n+2} x_{n+2}.$$

Therefore the centroids of all the r-faces lie on the n-sphere which passes through all the orthocentres of the same faces.

10. In the equation of this sphere take $r=n$, and it becomes

$$\sum_1^{n+1} \rho_k x_k + (n+1)\, \rho_{n+2} x_{n+2} = 0,$$

which is satisfied by the coordinates of P and G and has its centre on PG. The equation is therefore that of the n-sphere on PG as diameter.

The intersection of the n-sphere

$$\sum_1^{n+1} \rho_k x_k = (n-2r-1)\, \rho_{n+2} x_{n+2},$$

with the r-face

$$\rho_{r+2} x_{r+2} = \rho_{r+3} x_{r+3} = \ldots = \rho_{n+2} x_{n+2},$$

is given by

$$\sum_1^{r+1} \rho_k x_k + (n-r)\, \rho_{n+2} x_{n+2} = (n-2r-1)\, \rho_{n+2} x_{n+2},$$

or

$$\sum_1^{r+1} \rho_k x_k + (r+1)\, \rho_{r+2} x_{r+2} = 0.$$

It is therefore the r-sphere on the r-face, of which the line joining the orthocentre and centroid of that r-face is diameter. This is true whatever r-face be taken.

11. Special notice may be directed to that n-sphere of this series for which

$$n-2r-1=-1,$$

i.e.,

$$r=\tfrac{1}{2}n.$$

The equation then takes the form

$$\sum_{1}^{n+2}\rho_k x_k=0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15),$$

which is symmetrical with respect to all the $n+2$ vertices. Hence, if n be even, the n-sphere, which passes through the orthocentres (and centroids) of all $\frac{1}{2}n$-faces of a figure, of which $n+1$ of the given $n+2$ points are vertices and the remaining point the orthocentre, is identically the same in all the $n+2$ possible cases. This is the true analogue, in higher space of even dimensions, of the nine-point circle of plane geometry. The square of the radius of this n-sphere is

$$\sum_{1}^{n+2}\rho_k^2/(n+2)^2.$$

If the dimensions of the space are odd, this symmetrical n-sphere still exists, but there are no faces whose orthocentres and centroids lie upon it.

12. The form of the equation

$$\sum_{1}^{n+1}\rho_k x_k=(n-2r-1)\,\rho_{n+2}x_{n+2},$$

shows at once that all the n-spheres corresponding to different values of r, whether integral or not, have a common intersection with

$$x_{n+2}=0,$$

i.e., with the fundamental n-sphere at the orthocentre of the figure, which is itself an n-sphere of the series, viz., that for which $r=\infty$.

The common section of all these n-spheres is an $(n-1)$-sphere, and the distance of its centre T from P is determined by the relation

$$OT^2-PT^2=R^2-\rho_{n+2}^2,$$

whence

$$PG.PT=-\rho_{n+2}^2,$$

and the square of the radius of the common section is

$$\sum_1^{n+1} \rho_k^{\,2} \frac{\rho^2_{n+2}}{4l^2}.$$

The n-sphere of infinite radius belonging to the series is

$$\sum_1^{n+1} \rho_k x_k = (n+1)\,\rho_{n+2} x_{n+2} \ldots\ldots\ldots\ldots\ldots\ldots (16).$$

This radical $(n-1)$-space common to the series cuts any edge AB externally at Q in the same ratio as that in which it is divided internally by the projection of any vertex C upon it. There are $\frac{1}{2}n\,(n+1)$ such points Q on each of the $n+2$ sections (16); and n such sections pass through each of the $\frac{1}{2}\,(n+1)\,(n+2)$ points Q. For instance, in three dimensions the six points Q on the six edges of an orthocentric tetrahedron lie on a plane perpendicular to OP, viz., the radical plane of the circumsphere and other spheres represented by equation (11); and three of these planes pass through each of the ten points Q.

13. The equation to an inscribed or escribed n-sphere may now be determined. Let it be

$$\sum_1^{n+2} \alpha_k x_k = 0,$$

where the values of α_k have to be determined.

This n-sphere touches each $(n-1)$-face, viz.,

$$\rho_k x_k = \rho_{n+2} x_{n+2},$$

for all values of k from 1 to $n+1$, therefore

$$\alpha_k \rho_k - \alpha_{n+2} \rho_{n+2} = \pm a_k,$$

where a_k is the distance of the k^{th} vertex from the orthocentre.

Now $$\sum_1^{n+2} \alpha_k^{\,2} = \sum_1^{n+1} \frac{(\alpha_{n+2}\rho_{n+2} + a_k)^2}{\rho_k^{\,2}} + \alpha^2_{n+2}.$$

Therefore $$1 = 2\alpha_{n+2}\rho_{n+2} \sum_1^{n+2} \frac{a_k}{\rho_k^{\,2}} + \sum_1^{n+1} \left(1 + \frac{\rho^2_{n+2}}{\rho_k^{\,2}}\right)$$

$$= 2\alpha_{n+2}\rho_{n+2} \sum_1^{n+2} \frac{a_k}{\rho_k^{\,2}} + n.$$

But $$\frac{1}{\rho} = \sum_1^{n+2} \frac{\alpha_k}{\rho_k} = \sum_1^{n+2} \frac{a_k}{\rho_k^{\,2}}.$$

Therefore $2\alpha_{n+2}\rho_{n+2} + (n-1)\rho = 0.$

Therefore $2\alpha_k\rho_k = 2a_k - (n-1)\rho.$

Therefore the equation to the inscribed or escribed n-sphere is

$$\sum_1^{n+2}\left(\frac{a_k}{\rho_k}x_k\right) = \tfrac{1}{2}(n-1)\rho\sum_1^{n+2}\frac{x_k}{\rho_k} \quad \ldots\ldots\ldots\ldots(17),$$

where $1/\rho$ has the value already determined, viz., $\Sigma\frac{a_k}{\rho_k^2}$; and a_k is be taken with the appropriate sign for the particular n-sphere in question. The right-hand side of equation (17) is equal to $-(n-1)\rho$.

For the coordinates of the centre,

$$x_k' = \frac{\rho(\rho - 2\alpha_k\rho_k)}{\rho_k}$$

$$= \frac{\rho(n\rho - 2a_k)}{\rho_k}.$$

Hence, if o is the centre of any inscribed or escribed n-sphere,

$$Po^2 = n\rho^2 + \rho_{n+2}^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(18).$$

14. In plane geometry, the condition that the nine-point circle, viz.,

$$\Sigma\rho_k x_k = 0$$

should touch any inscribed or escribed circle

$$\Sigma\alpha_k\rho_k = 0,$$

is that $(\Sigma\alpha_k\rho_k)^2 = \Sigma\rho_k^2.$

Now $(\Sigma\alpha_k\rho_k)^2 = (\Sigma a_k - 2\rho)^2$

$$= \left\{(a_1+a_2+a_3)\left(\frac{a_1}{\rho_1^2}+\ldots\right) - 2\right\}^2\rho^2$$

$$= \left\{a_1a_2\left(\frac{1}{\rho_3^2}+\frac{1}{\rho_4^2}\right)+\ldots\right\}\rho^2$$

$$= \frac{a_1^2a_2^2a_3^2}{\rho_4^4} = \Sigma\rho_k^2.$$

Therefore the nine-point circle touches the sixteen inscribed and escribed circles belonging to the four triangles of an orthocentric quadrangle.

POLYMORPHISM OF $(x^\beta y^n \sim j x^\alpha y^m)$.

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[The author is indebted to Mr. H. J. Woodall, A.R.C.Sc, for help in reading the proof-sheets.]

1. *Introduction.* It is proposed to investigate the solution $(x, y\,;\ x', y'\,;$ &c.) of

$$N = \phi(x, y) = \phi(x', y') = \phi(x'', y'') = \text{ &c.} \quad \ldots\ldots(1a),$$

$$= \phi`(x, y) = \phi`(x', y') = \text{ &c.} \quad \ldots\ldots(1b),$$

where—

$$\phi(x, y) = x\beta y^n - j x^\alpha y^m \quad \ldots\ldots(2a),$$

$$\phi`(x, y) = j x^\alpha y^m - x^\beta y^n \quad \ldots\ldots(2b),$$

$j = \pm 1$, the same throughout all ϕ, or all $\phi`$.

β, α, m, n integers; $\beta > \alpha$; $m > n$.

Each pair of x, y mutually prime.

Now let $\psi(x, y)$ denote either $\phi(x, y)$ or $\phi`(x, y)$.....(3), and let G denote "Greatest Common Measure."

Then

$$\psi(x, y) \equiv 0 \pmod{x^\alpha y^n} \quad \ldots\ldots(4a),$$

$$\psi(x, y) = \psi(x', y') \equiv 0 \pmod{G \text{ of } x^\alpha y^n,\ x'^\alpha y'^n} \quad \ldots\ldots(4b),$$

$$\psi(x, y) = \psi(x', y') = \psi'(x'', y'') \equiv 0 \pmod{G \text{ of } x^\alpha y^n,\ x'^\alpha y'^n, \text{ &c.}} \quad \ldots\ldots(4c),$$

and so on.

The proposed equation in its most general form $(1a, b)$ seems too difficult for a really general solution. Solutions of considerable interest may however be obtained with one of the assumptions—

$$x = x' = x'' = \text{ &c.}\,; \quad y \neq y' \neq y'' \neq \text{ &c.} \quad \ldots\ldots(5a),$$

$$y = y' = y'' = \text{ &c.}\,; \quad x \neq x' \neq x'' \neq \text{ &c.} \quad \ldots\ldots(5b).$$

The results of either assumption $(5a, b)$ are derivable from the other by simply interchanging $(x, x', x'', \ldots, \beta, \alpha)$ with $(y, y', y'', \ldots, m, n)$. It will suffice therefore to use only one of them: the former $(5a)$ is used throughout this Paper.

The Paper is divided into two Parts (I., II.):

Part I., Art. 2—12 deals entirely with the form $\phi(x, y)$.

Part II., Art. 13—18 deals with both forms $\phi(x, y)$, $\phi`(x, y)$.

Part I. [Form $\phi(x, y)$].

2. *Dimorphism.* With the assumption (5*a*), the dimorphism

$$N = \phi(x, y) = \phi(x', y') \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6),$$

requires, and is given by

$$x^{\beta-\alpha} = j \cdot \frac{y^m - y'^m}{y^n - y'^n} = j \cdot \chi(y, y'), \text{ suppose} \ldots\ldots\ldots\ldots\ldots\ldots (7).$$

If this can be solved so as to give x an integer, every solution (x, y, y') gives one Dimorph (6).

The quantity $\chi(y, y')$ being always +, it follows that

When $j = +1$, one solution may be possible for each pair of $y, y' \ldots\ldots (8a)$.

When $j = -1$, and $\beta - \alpha$ is *odd*, one solution may be possible for each pair of $y, y' \ldots (8b)$.

When $j = -1$, and $\beta - \alpha$ is *even*, then x is *irrational; [no solution]...(8*c*).

The dimorph N of (5) may be reduced to

$$N = jx^\alpha y^n y'^n \cdot \frac{y^{m-n} - y'^{m-n}}{y^n - y'^n} = jx^\alpha y^n y'^n \cdot \varsigma(y, y'), \text{ suppose} \ldots\ldots\ldots (9).$$

It often happens that the expression (7) for $x^{\beta-\alpha}$ is expressible (arithmetically) in more than one way in that form, so that

$$x^{\beta-\alpha} \text{ may } = j \cdot \chi_1(y_1, y_1') = j \cdot \chi_2(y_2, y_2'), \text{ \&c} \ldots\ldots\ldots\ldots\ldots\ldots (10),$$

where the pairs (y_1, y_1'), (y_2, y_2'), &c., are usually *all different.*

In this case each such expression $\chi(y_r, y_r')$ yields in general a *single dimorph* N, so that

$$N_1 = \phi(x, y_1) = \phi(x, y_1'); \quad N_2 = \phi(x, y_2) = \phi(x, y_2') \ldots\ldots\ldots\ldots (11),$$
$$N_3 = \text{ \&c.}$$

but, in general $N_1 \neq N_2 \neq N_3 \neq$ &c. $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$,

so that Trimorphism of N *does not result in this way.*

3. *Trimorphism.* With the assumption (5*a*), the trimorphism—

$$N = \phi(x, y) = \phi(x, y') = \phi(x, y'') \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13),$$

requires, and is given by

$$x^{\beta-\alpha} = j \cdot \chi(y, y') = j \cdot \chi(y', y'') = j \cdot \chi(y'', y) \ldots\ldots\ldots\ldots\ldots\ldots (14).$$

where χ has the meaning shown in the Result (7).

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* These Cases—(insoluble by the assumption $x = x'$ of this Paper)—are much more difficult.

Any two of these involve the third. This result (14) differs from the previous result (9) in that only three values of y appear in the trimorph.

The quantities χ being all +, the Results (8*a*, *b*, *c*), limiting the use of $j=-1$, obtain here also.

If (14) can be solved so as to give x an integer, every solution (x, y, y', y'') gives one Trimorph (13).

Using the symbol ϑ defined in (9), N may now be written in the three forms

$$N=jx^\alpha y^n y'^n . \vartheta(y, y') = jx^\alpha y'^n y''^n . \vartheta(y', y'') = jx^\alpha y''^n y^n . \vartheta(y'', y) \ldots (15),$$

and also in the form

$$N = jx^\alpha y^n y'^n y''^n . \Theta(y, y', y'') \ldots (15a),$$

where
$$\Theta(y, y', y'') = \frac{\vartheta(y', y')}{y''} = \frac{\vartheta(y', y'')}{y} = \frac{\vartheta(y'', y)}{y'} \ldots (15b).$$

These three* expressions are equal in virtue of the preceding results.

It may happen that the expression (14) for $x^{\beta-\alpha}$ is expressible in more than one way in a set of three forms χ, connected as in (14), in terms of three quantities (y_r, y_r', y_r''), the triplets being usually all different. In this case each such triplet (y_r, y_r', y_r'') yields a single Trimorph N. And these trimorph N are usually unequal, so that Tetramorphism *does not result in this way.*

3*a*. *Conditions of Trimorphism.* This appears to depend on the property peculiar to all Cubans

$$\frac{z^3-z'^3}{z-z'} = \frac{z''^3+z^3}{z''+z} = \frac{z''^3+z'^3}{z''+z'}, \text{ identically, } [\text{when } z''=z+z'] \ldots (16).$$

To find such numbers, take

$$X = \text{any integer of form } (a^2+3b^2), \ [a \text{ prime to } b] \ldots (17).$$

Then X^κ is also of same form

$$X^\kappa = A^2+3B^2, \ [A \text{ prime to } B] \ldots (18).$$

where A, B can be found from a, b (supposed known) by known Rules, *e.g.*

$$\kappa=2 \text{ has } A=a^2-3b^2, \quad B=2ab \ldots (18a),$$
$$\kappa=3 \text{ has } A=a^3-9ab^2, \quad B=3b(a^2-b^2) \ldots (18b).$$

* It does not seem easy to express these in terms of a single explicit function of y, y', y''.

To express X^κ in the above three forms (16), take

z'' = the greatest of $A+B$, $2B$; $z = A \sim B$ ..... (19*a*),

z' = the lesser of $A+B$, $2B$ ..... (19*b*).

When X is prime, or a prime-power, then (a, b), (A, B) and the triplet z, z', z'' are *unique*. But when X is composite, it will have more than one expression as $X = a^2 + 3b^2$. Each such expression (a, b) yields one set of (A, B) and one triplet (z, z', z''). And each set of (a, b), (A, B), (z, z', z'') is generally *different from all the rest* (though they have the same X and X^κ).

4. *Solvable Cases.* The production of actual Dimorphs and Trimorphs (N) requires the solution of Eq. (7) or (14) respectively in integers. This can be done in the following Cases, depending on the relations of β, α, m, n.

I. *Algebraic Cases.* 1°. $\beta - \alpha = 1$, $n = 1$; 2°. $\beta - \alpha = 1$, $m = kn$.

II. *Dimorphs.* 1°. $m = 2$, $n = 1$; 2°. $m = 4$, $n = 2$.

III. *Trimorphs.* 1°. $m = 3$, $n = 1$; 2°. $m = 3$, $n = 2$.

IV. *Small* $\beta - \alpha, m, n$. 1°. $\beta - \alpha = 2$, $m = 4$, $n = 1$.

The detailed solution of the above is given in Art. 5—12 with examples in several Cases.

5. I. *Algebraical Cases* $[\beta - \alpha = 1]$. Two Cases (1°, 2°) are algebraically solvable—

6. Case I.—1°. $\beta - \alpha = 1$, $n = 1$,

$$N = x^{\alpha+1} y - j x^\alpha y^m = x'^{\alpha+1} y' - j x'^\alpha y'^m \quad \ldots\ldots (20),$$

Here, by (7),

$$x = j \cdot \frac{y^m - y'^m}{y - y'}, \text{ always an integer} \quad \ldots\ldots (21).$$

Here any pair of y, y' give an integer x, and give rise to a single Dimorph N, which reduces by (9) to

$$N = j x^\alpha y y' \cdot \frac{y^{m-1} - y'^{m-1}}{y - y'}, \text{ (an integer)} \quad \ldots\ldots (22).$$

When $m = 3$ in above, this Case yields a Trimorph N: this is fully dealt with in Art. 11 below.

7. Case I.—2°. $\beta - \alpha = 1$, $m = kn$,

$$N = x^{\alpha+1} y^n - j x^\alpha y^{kn} = x'^{\alpha+1} y'^n - j x'^\alpha y'^{kn} \quad \ldots\ldots (23).$$

Here, by (7)

$$x = j.\frac{y^{kn} - y'^{kn}}{y^n - y'^n}, \text{ always an integer} \quad \ldots\ldots(24).$$

Here any pair of y, y' give an integer x, and give rise to a single Dimorph N, which reduces by (9) to

$$N = jx^\alpha y^n y'^n . \frac{y^{kn-n} - y'^{kn-n}}{y^n - y'^n}, \text{ (an integer)} \quad \ldots\ldots(25).$$

Here Case I.—1° is only a special Case of I.—2° (with $n = 1$): and Case I.—2° may be treated under Case I.—1° by writing $y^n = Y$, $y'^n = Y'$.

When $k = 3$, so that $m = 3n$, a single Trimorph N results. This may be treated as a special Case of Art. 11 below, by writing $\beta - \alpha = 1$ therein, and generalising it by writing also $y^n = Y$, $y'^n = Y'$ therein.

8. II., III. *Small values of* m, n. The following cases, with small m, n, are solvable for all values of $(\beta - \alpha)$.

II. *Dimorphs.* 1°. $m = 2$, $n = 1$; 2°. $m = 4$, $n = 2$.

III. *Trimorphs.* 1°. $m = 3$, $n = 1$; 2°. $m = 3$, $n = 2$.

9. Case II.—1°. $m = 2$, $n = 1$; *Dimorph N.*

$$N = x^\beta y - jx^\alpha y^2 = x'^\beta y' - jx'^\alpha y'^2 \quad \ldots\ldots(26).$$

Here, by (7)

$$x^{\beta-\alpha} = j(y + y') \quad \ldots\ldots(27).$$

Here x may be any integer. Every pair of y, y' whose sum satisfies (27) gives a single Dimorph N, which reduces by (9) to

$$N = jx^\alpha yy' \quad \ldots\ldots(28).$$

10. Case II.—2°. $m = 4$, $n = 2$; *Dimorph N.*

$$N = x^\beta y^2 - jx^\alpha y^4 = x'^\beta y'^2 - jx'^\alpha y'^4 \quad \ldots\ldots(29).$$

Here, by (7)

$$x^{\beta-\alpha} = j(y^2 + y'^2) \quad \ldots\ldots(30).$$

Here x may be taken = any integer of form $x = j(\eta^2 + \eta'^2)$.

Then $x^{\beta-\alpha}$ will also be of form $j(y^2 + y'^2)$, where y, y' may be found by known Rules from η, η' (supposed known). When x is a prime, or a prime-power, this form will be unique, and will yield a single Dimorph N, which is reducible by (9) to

$$N = jx^\alpha y^2 y'^2 \quad \ldots\ldots(31).$$

When x is composite, it will have more than one form $x = j(\eta^2 + \eta'^2)$, each of which will yield one pair of (y, y'), and a single Dimorph N. And these products will be unequal.

Ex.	$\beta - \alpha$;	x;	η, η';	y, y';	N.
	1	5	1, 2	1, 2	$j.5^\alpha.(1.2)^2$
	2	5	1, 2	3, 4	$j.5^\alpha.(3.4)^2$
	3	5	1, 2	11. 4	$j.5^\alpha.(11.4)^2$

11. Case III.—1°. $m = 3$, $n = 1$. *Trimorph* N.

$$N = x^\beta y - j x^\alpha y^3 = x'^\beta y' - j x'^\alpha y'^3 = x''^\beta y'' - j x''^\alpha y''^3 \ldots\ldots\ldots\ldots (32).$$

Here, by (7) and Art. 3*a*

$$x^{\beta-\alpha} = j\frac{y^3 - y'^3}{y - y'} = j\frac{y''^3 + y^3}{y'' + y} = j\frac{y''^3 + y'^3}{y'' + y'}, \quad [y'' = y + y'] \ldots\ldots\ldots (33).$$

Here x may be assumed any integer of form $x = a^2 + 3b^2$, whereby $x^{\beta-\alpha}$ is of form $x^{\beta-\alpha} = A^2 + 3B^2$, where A, B may be found from a, b by known Rules, see Art. 3*a*; and y, y', y'' are those given by

$y'' =$ the greatest of $A + B$, $2B$; $y = A \sim B$.....(34*a*),

$y' =$ the lesser of $A + B$, $2B$, [y, y' interchangeable]....(34*b*).

When x is a prime, or a prime-power, this gives (a, b), (A, B) unique, and one triplet y, y, y'', and thence a single Trimorph N, which by (9) is reducible to

$$N = -j k^\alpha y y' y''; \quad [y'' \text{ is to be taken negative}] \ldots\ldots\ldots (35).$$

When x is composite, it will have more than one form of $x = a^2 + 3b^2$. Each such form will yield one pair of (A, B), and thereby one triplet (y, y', y''), and thence a single Trimorph N. Each set of (a, b), (A, B), (y, y', y'') is generally different from all the rest; and the products N are all different, though arising from the same $x^{\beta-\alpha}$.

Ex.	$\beta - \alpha$;	x;	a, b;	A, B;	y, y', y'';	N.
	1	7	2, 1	2, 1	2, 1, −3	$j.7^\alpha.(2.1.3)$
	2	7	2, 1	1, 4	3, 5, −8	$j.7^\alpha.(3.5.8)$
	3	7	2, 1	10, 9	1, 18, −19	$j.7^\alpha.(1\,18.19)$

Positive integer solution. When $\beta = m = 3$, $\alpha = n = 1$, the above solution may be converted into one in + integers by simply interchanging the *numerical* values (disregarding sign) of x, y'' above given in the Form $\phi(x'', y'')$.

11*a*. Case III.—2°. $m=3,\ n=2$. *Trimorph N.*

$$N=x^\beta y^2-jx^\alpha y^3=x'^\beta y'^2-jx'^\alpha y'^3=x''^\beta y''^2-jx''^\alpha y''^3.$$

Here, by (7)

$$x^{\beta-\alpha}=j\cdot\frac{y^3-y'^3}{y^2-y'^2}\qquad\qquad(36).$$

Writing now $y=\eta(\eta+\eta'),\ y'=\eta'(\eta+\eta'),\ y''=-\eta\eta'$...(37),

Result (36) reduces—by Art. 34—to the Trimorph form :

$$x^{\beta-\alpha}=j\cdot\frac{\eta^3-\eta'^3}{\eta-\eta'}=j\cdot\frac{\eta''^3+\eta^3}{\eta''+\eta}=j\cdot\frac{\eta''^3+\eta'^3}{\eta''+\eta'},\ [\eta''=\eta+\eta'].\ldots(38).$$

Here x may be assumed any integer of form $x=a^2+3b^2$ (with a prime to b), whereby (as in Art. 3*a*) $x^{\beta-\alpha}$ is of form $x^{\beta-\alpha}=A^2+3B^2$, where A, B may be found from a, b by known Rules, and η, η', η'' are then given by

$$\eta''=\text{the greater of } A+B,\ 2B;\ \ \eta=A-B\qquad\qquad(39a),$$

$$\eta'=\text{the lesser of } A-B,\ 2B;\ [\eta,\eta' \text{ interchangeable}]\qquad\qquad(39b),$$

and y, y', y'' are then given by (37).

When x is prime, or a prime-power, this gives $(a,\ b)$, $(A,\ B)$ unique, and thereby one triplet $(\eta,\ \eta',\ \eta'')$ and one triplet $(y,\ y',\ y'')$ each unique, and thence a single Trimorph N, which by (9) is reducible to

$$N=j\cdot x^\alpha\cdot\frac{y^2y'^2}{y+y'}=j\cdot x^\alpha\cdot(\eta\eta'\eta'')^2\qquad\qquad(40).$$

When x is composite, it will have more than one form of $x=a^2+3b^2$. Each such form will yield one pair of $(A,\ B)$, and thereby one triplet $(\eta,\ \eta',\ \eta'')$ and one triplet $(y,\ y',\ y'')$. Each set of $(a,\ b)$, $(A,\ B)$, $(\eta,\ \eta',\ \eta'')$, $(y,\ y',\ y'')$ is generally different from all the rest ; and the products N are all different, though arising from the same $x^{\beta-\alpha}$.

Ex. $\beta-\alpha$;	x;	$a,\ b$;	$A,\ B$;	$\eta,\ \eta',\ \eta''$;	$y,\ y',\ y''$;	N.
1	7	2, 1	2, 1	1, 2, 3	3, 6, −2	$j.7^\alpha.(1.2.3)^2$
2	7	2, 1	1, 4	3, 5, 8	24, 40, −15	$j.7^\alpha.(3.5.8)^2$
3	7	2, 1	10, 9	1, 18, 19	19, 342, −18	$j.7^\alpha.(1.18.19)^2$.

12. Case IV. *Small values of* $\beta-\alpha$, m, n.

One more Case below with $\beta-\alpha=2$ is also solvable.

Case IV. $\beta-\alpha=2$; $m=4,\ n=1$; *Dimorph N.*

$$N=x^{\alpha+2}y-x^\alpha y^4=x'^{\alpha+2}y'-x'^\alpha y'^4;\ [j \text{ must } =+1]\qquad\qquad(41).$$

Here, by (7)

$$x^2=(y+y')(y^2+y'^2) \quad \ldots\ldots(42),$$

Take y, y' any integers subject to the conditions

$$y+y'=z^2, \quad y^2+y'^2=Z^2, \text{ so that } x=zZ \ldots\ldots(43).$$

Then y, y', Z must be of forms

$$y=t^2\sim u^2, \quad y'=2tu, \quad Z=t^2+u^2 \quad \ldots\ldots(43a),$$

whence $$z^2=y+y'=(t+u)^2-2tu.$$

Now take z any quantity of form $x=e^2-2f^2$ (with e prime to f).

Then $x^2=E^2-2F^2$, where $E=e^2+2f^2$, $F=2ef$, [E prime to F].

$$t+u=E, \quad u=F, \quad t=E-F \ldots\ldots(44).$$

Hence every number z of form $z=e^2-2f^2$ yields a pair (E, F); and this pair (E, F) has an infinite series of automorphs (E_1, F_1), (E_2, F_2), &c., such that

$$z^2=E^2-2F^2=E_1^2-2F_1^2=E_2^2-2F_2^2=\text{ &c. } [E_r \text{ prime to } F_r] \ldots(45).$$

Each automorph pair (E_r, F_r) yields one pair (t, u) by (44), and one set (y, y', Z) by (43), and a single x by (41), and finally a single Dimorph N, which by (9) reduces to

$$N=x^\alpha y y'(y^2+yy'+y'^2) \ldots\ldots(46).$$

and the set of numbers (t, u, y, y', Z, x) arising from any one pair (E_r, F_r) are different to those arising from all the other automorphs E_r, and the Dimorph products N are generally all unequal.

When z is a prime, or a prime-power, the series of (E, F) is unique, and the series of (t, u, y, y', x, N) is unique.

But when z is composite, it has more than one basic form $z=e^2-2f^2$ (not mutually automorphs). Each such basic form (e, f) has its own infinite series of automorphs (E_r, F_r), and each automorph yields one set of (t, u, y, y', Z, x), and finally one Dimorph N. And these sets are in general all different, and the products N all unequal, not only those of one series, but also those of different series.

Ex. $\beta-\alpha$;	z;	e, f;	E, F;	t, u;	y, y';	Z;	x	N.
2	7	3, 1	9, 4	5, 4	9, 40	41	7.41	x^α.(9.40.2041).
2	7	3, 1	11, 6	5, 6	11, 60	61	7.61	x^α.(11.60.4381).

Part II. [Forms $\phi(x, y)$, $\phi'(x, y)$].

13. *Pseudo-Dimorphism.* The case of what may be called Pseudo-Dimorphism of N will now be considered, where

$$N=x^\beta y^n-x^\alpha y^m=x'^\alpha y'^m-x'^\beta y'^n \quad [\beta>\alpha, \ m>n] \ldots\ldots(47).$$

The symbol j is not here introduced (as in Part I.) ; because, if both signs be + (when $j=-1$), the case falls under Part I.

The case of m, n *both odd* may also be excluded : because by writing $-y'$ for y' this Case is brought under Part I. It remains to consider the case when mn is *even*.

The assumption will be made again (as in Part I.)

$$x=x', \quad y \neq y' \ldots\ldots\ldots\

Then, by (52)

$$x = y^m + y'm \quad \ldots\ldots(56).$$

Here any pair of integers y, y' give an integer x, and thereby a single Pseudo-Dimorph N, which by (50) reduces to

$$N = x^\alpha yy'(y'^{m-1} - y^{m-1}) \quad \ldots\ldots(57).$$

17. Case II.—2°. $\beta - \alpha = 2$ or 4, $m = 2$, $n = 1$; *Pseudo-Dimorph N.*

$$N = x^\beta y - x^\alpha y^2 = x'^\alpha y'^2 - x'^\beta y' \quad \ldots\ldots(58).$$

Take $y + y' = 1 \quad \ldots\ldots(59).$

By (49) this gives

$$x^{\beta-\alpha} = y^2 + y'^2 = 2y^2 - 2y + 1 \quad \ldots\ldots(60).$$

Here x must be of form $x = \eta^2 + \eta'^2$, and $\beta - \alpha$ must be such that $x^{\beta-\alpha}$ yields values of y, y' satisfying the condition (60). This is possible when $\beta - \alpha = 2$ or 4. Each pair of suitable y, y' yield one x, and a single Pseudo-Dimorph N, which is reducible by (50) to

$$N = x^\alpha yy'(y' - y) \quad \ldots\ldots(61),$$

Ex.	$\beta-\alpha$;	y, y';	x;	N.
	2	−3, 4	5	$-5^\alpha.(3.4.7)$
	2	−20, 21	29	$-29^\alpha.(20.21.41)$
	4	−119, 120	13	$-13^\alpha.(119.120.239)$

18. Case III. $\beta - \alpha = 1$, $m = 6$, $n = 2$; *Pseudo-Trimorph N.*

$$N = x^{\alpha+1}y^2 - x^\alpha y^6 = x'^{\alpha+1}y'^2 - x'^\alpha y'^6 = x''^\alpha y''^6 - x''^{\alpha+1}y^2 \quad \ldots\ldots(62).$$

Writing the three forms N, N', N'', it will be seen that $N = N'$ is Dimorph, whilst $N = N''$ and $N' = N''$ are Pseudo-Trimorph. Here, by formula (7) and (48), and Art. 3*a*,

$$x = \frac{y^6 - y'^6}{y^2 - y'^2} = \frac{y''^6 + y^6}{y''^2 + y^2} = \frac{y''^6 + y'^6}{y''^2 + y'^2}, \text{ [where } y''^2 = y^2 + y'^2] \quad \ldots\ldots(63).$$

Herein x is always an integer: and the triplet y, y', y'' gives N in the three forms (62) above, Pseudo-Trimorph, and by (50) N reduces to

$$N = x^\alpha y^2 y'^2 y''^2 \quad \ldots\ldots(64).$$

Ex.	y, y', y'';	x;	N.
	3, 4, 5	481	$x^\alpha.(3.4.5)^2$
	5, 12, 13	24961	$x^\alpha.(5.12.13)^2$

Appendix.

Euler's Case. The following Dimorph

$$N = x^3y + y^3x = x'^3y' + y'^3x',$$

cannot be solved by the assumption $x = x'$ used in this Paper, because here $j = -1$, and $\beta - \alpha = 2$, give x irrational [see Art. 2 (8*c*)]. But, writing

$$x = X + Y, \quad y = X - Y, \quad x' = X' + Y', \quad y' = X' - Y',$$

gives

$$N = 2(X^2 - Y^2)(X^2 + Y^2) = 2(X^2 - Y'^2)(X'^2 + Y'^2),$$

whence

$$X^4 - Y^4 = X'^4 - Y'^4,$$

an equation solved by *Euler. The lowest known solution is given below—

$X,\ Y;$	$X',\ Y';$	$x,\ y;$	$x',\ y'.$
134, 59;	158, 132;	193, 75;	291, 25

whence

$$N = 2.3.25.13.17.97.193.$$

A RELATION BETWEEN THE CIRCLES ASSOCIATED WITH A TRIANGLE.

By *T. C. Lewis.*

Let R, r', r, r_1, r_2, r_3 be the radii of circumscribing, nine-point, inscribed, and escribed circles respectively.

Let ρ_1, ρ_2, ρ_3, ρ_4 be the radii of the four circles, each cutting the other three at right angles, and having their centres at the vertices and orthocentre of the triangle.

Let a_1, a_2, a_3 be the lengths of the straight lines from the orthocentre to the three vertices.

It is known that

$$\frac{1}{\rho_1^2} + \frac{1}{\rho_2^2} + \frac{1}{\rho_3^2} + \frac{1}{\rho_4^2} = 0,$$

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* Euler's *Comment. Arithm*, Petrop., 1849; t. i., pp. 473—476; t. ii., pp. 281—293; t. ii., pp. 450—456.

For an Abstract of Euler's Solution, with detail of the results, and a Table of numerical examples, see the author's Paper on *Diophantine Factorisation of Quartans*, Art. 27—27*d*, in *Messenger of Mathematics*., vol. xxxviii. (1908), pp. 83—86.

that $$4R^2 = 16r'^2 = \rho_1^2 + \rho_2^2 + \rho_3^2 + \rho_4^2,$$

and that $$a_1^2 = \rho_1^2 + \rho_4^2, \text{ \&c.}$$

Equating two expressions for the square of the area of the triangle, we have

$$rr_1r_2r_3 = -\frac{\rho_1^2\rho_2^2\rho_3^2}{4\rho_4^2};$$

therefore

$$r^2r_1^2r_2^2r_3^2\rho_4^4(\rho_1^2 + \rho_2^2 + \rho_3^2 + \rho_4^2) = \rho_1^4\rho_2^4\rho_3^4r'^2;$$

therefore $$r^2r_1^2r_2^2r_3^2a_1^2a_2^2a_3^2 = \rho_1^4\rho_2^4\rho_3^4r'^2;$$

therefore $$\tfrac{1}{2}R = r' = \frac{a_1a_2a_3r_1r_2r_3}{\rho_1^2\rho_2^2\rho_3^2}r.$$

Here a_1, a_2, a_3 are affected by the same sign as ρ_1^2, ρ_2^2, ρ_3^2 respectively.

THE EIGHT QUEENS PROBLEM.

By *Thorold Gosset.*

In the *Messenger of Mathematics*, vol. xxxix., p. 19, Mr. G. T. Bennett has proved that it is impossible to combine on the same board more than six sets of queens satisfying the conditions of the problem.

A much simpler proof of this may be of interest. In the 12 fundamental solutions

$$\dot{4}6\dot{1}52\dot{8}3\dot{7}, \quad \dot{6}1\dot{5}28\dot{3}\dot{7}\dot{4}, \quad \dot{3}5\dot{8}41\dot{7}\dot{2}6, \quad \dot{5}8\dot{4}17\dot{2}\dot{6}3,$$

$$\dot{7}2\dot{6}3\dot{1}\dot{4}8\dot{5}, \quad \dot{5}\dot{7}26\dot{3}\dot{1}\dot{4}8, \quad \dot{5}\dot{7}26\dot{3}\dot{1}\dot{8}\dot{4}, \quad 1\dot{6}8\dot{3}7\dot{4}\dot{2}\dot{5},$$

$$\dot{4}8\dot{1}57\dot{2}\dot{6}3, \quad \dot{5}\dot{1}\dot{4}\dot{6}8\dot{2}\dot{7}3, \quad \dot{4}\dot{2}\dot{7}\dot{5}\dot{1}\dot{8}\dot{6}3, \quad \dot{3}\dot{5}\dot{2}\dot{8}\dot{1}\dot{7}\dot{4}\dot{6},$$

I have dotted those numbers which indicate that a queen stands on one of the 20 squares forming a symmetrical octagon composed of the 4 central squares along each of the 4 sides of the board and the 4 squares diagonally adjacent to corner squares.

The proof is as follows:—Every fundamental solution, and therefore by symmetry every solution, has at least 3 queens on the 20 squares of the octagon; consequently more than 6 solutions on the same board would involve at least 21 queens on 20 squares.

In all the above solutions there are 44 queens on the 20 squares in question, so that 12 different fundamental solutions could not be combined on two boards.

THEOREMS IN OBLIQUE COORDINATES.

By *Prof. Anglin*, University College, Cork.

THIS paper is a sequel to that entitled "Expressions for the volume of a tetrahedron," which appeared in the *Messenger of Mathematics* (vol. xli., p. 91), and contains results which, I believe, have not hitherto been recorded.

1. To obtain an expression for any plane area, corresponding to that given by

$$\Delta^2 = \Delta_x^{\,2} + \Delta_y^{\,2} + \Delta_z^{\,2}$$

in rectangular axes, where Δ_x, Δ_y, Δ_z are the areas of the projections of Δ on the coordinate planes.

We shall first find the area of the triangle ABC, which is the base of a tetrahedron vertex O, in terms of the areas of the faces intersecting in O and the angles between their planes.

If the edges meeting in O be a, b, c, and be inclined to one another at angles α, β, γ respectively, then substituting

$$x^2 = b^2 + c^2 - 2bc\cos\alpha$$

with corresponding expressions for y^2, z^2 in the relation

$$16\Delta^2 = 4y^2z^2 - (y^2 + z^2 - x^2)^2,$$

where Δ is the area of the triangle whose sides are x, y, z, we shall get

$$\begin{aligned} 4\Delta^2 &= (c^2 + a^2 - 2ca\cos\beta)\,(a^2 + b^2 - 2ab\cos\gamma) \\ &\qquad - (a^2 + bc\cos\alpha - ca\cos\beta - ab\cos\gamma)^2 \\ &= \Sigma\,(bc\sin\alpha)^2 - 2abc\,.\,\Sigma a\,(\cos\alpha - \cos\beta\cos\gamma) \\ &= \Sigma\,(bc\sin\alpha)^2 - 2\Sigma\,(ca\sin\beta\,.\,ab\sin\gamma\,.\cos\lambda), \end{aligned}$$

since $$\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos\lambda,$$

where λ is the angle between the planes intersecting in OA. Hence

$$\Delta^2 = \Sigma\Delta_1^{\,2} - 2\Sigma\Delta_2\Delta_3\,.\cos\lambda \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where Δ_1, Δ_2, Δ_3 are the areas of the faces of the tetrahedron intersecting in O, and λ, μ, ν the angles between their planes.

Now taking any triangle PQR (the coordinates of whose vertices are given), if the plane of PQR meet the coordinate axes in A, B, C respectively, since the areas of the triangles PQR and ABC are as the areas of their oblique projections on the coordinate planes respectively, we have

$$\frac{\text{area } PQR}{\text{area } ABC} = \frac{\Delta_x}{\Delta_1} = \frac{\Delta_y}{\Delta_2} = \frac{\Delta_z}{\Delta_3}, \text{ where } \Delta_x = \tfrac{1}{2}\begin{vmatrix} y_1, & z_1, & 1 \\ y_2, & z_2, & 1 \\ y_3, & z_3, & 1 \end{vmatrix} \sin\alpha,$$

with corresponding expressions for Δ_y and Δ_z, α, β, γ being the angles between the coordinate axes. Thus

$$\frac{(PQR)^2}{(ABC)^2} = \frac{\Sigma\Delta_x^2 - 2\Sigma\Delta_y.\Delta_z.\cos\lambda}{\Sigma\Delta_1^2 - 2\Sigma\Delta_2\Delta_3.\cos\lambda},$$

where λ, μ, ν are the angles between the coordinate planes. But the denominators are equal, therefore the numerators are equal; that is

$$(\Delta PQR)^2 = \Sigma\Delta_x^2 - 2\Sigma\Delta_y\Delta_z.\cos\lambda \ldots\ldots\ldots (2).$$

This result obviously holds for *any* plane area, where Δ_x, Δ_y, Δ_z are the areas of its oblique projections on the coordinate planes.

2. To find the length of the perpendicular from the point (x', y', z') on the plane $ax + by + cz + d = 0$.

We shall first find the perpendicular on the plane from the origin. If the plane meet the coordinate axes in A, B, C respectively, and p be the perpendicular on it from the origin O, we have

$$p(\Delta ABC) = 3 \text{ vol. tetrahedron } OABC = n.OA.OB.OC,$$

where

$$4n^2 = 1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma,$$

α, β, γ being the angles between the axes. Hence, by (1) we get

$$p\{\Sigma(OB.OC\sin\alpha)^2 - 2\Sigma(OA.OC\sin\beta.OA.OB\sin\gamma.\cos\lambda)\}^{\frac{1}{2}} = 2n.OA.OB.OC,$$

which, on multiplying by $d/OA.OB.OC$, gives

$$p\{\Sigma(d/OA)^2\sin^2\alpha - 2\Sigma(d^2/OB.OC)\sin\beta\sin\gamma.\cos\lambda\}^{\frac{1}{2}} = 2nd.$$

But $-d = a.OA = b.OB = c.OC$, so that

$$p\,(\Sigma a^2 \sin^2\alpha - 2\Sigma bc \sin\beta \sin\gamma . \cos\lambda)^{\frac{1}{2}} = 2nd.$$

Now, since the equation to the plane through (x', y', z') parallel to the given plane is

$$a\,(x-x') + b\,(y-y') + c\,(z-z') = 0,$$

it follows that the perpendicular from (x', y', z') on the given plane, being equal to the difference of the perpendiculars from O on it and the parallel plane through (x', y', z'), is equal to

$$2n\,(ax' + by' + cz' + d)\,/\,(\Sigma a^2 \sin^2\alpha - 2\Sigma bc \sin\beta \sin\gamma . \cos\lambda)^{\frac{1}{2}}.$$

Cor. The volume of a tetrahedron, in terms of the coordinates of its vertices, may be deduced from the foregoing results by writing down the equation to the plane through three of the vertices; when we shall get

$$6 \text{ vol.} = \begin{vmatrix} x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \\ x_4, & y_4, & z_4, & 1 \end{vmatrix} 2n.$$

3. To obtain corresponding results in homogeneous coordinates. If $ABCD$ be the tetrahedron of reference, we change to oblique Cartesians with D as origin and DA, DB, DC as axes. Thus, if x, y, z, u be the tetrahedral and x', y', z' the Cartesian coordinates of any point, we substitute $x' = ax$, $y' = by$, $z' = cz$ in the results involving Cartesians, where $DA = a$, $DB = b$, $DC = c$.

Let λ, μ, ν be the angles between the planes intersecting in DA, DB, DC respectively, and λ', μ', ν' those between the planes intersecting in BC, CA, AB respectively; then if A, B, C, D be the areas of the faces of the tetrahedron of reference, we observe that

$$D = A\cos\lambda' + B\cos\mu' + C\cos\nu', \text{ \&c.},$$

since $D = \Delta ABC =$ sum of projections on ABC of the three faces intersecting in D, and

$$D^2 = \Sigma A^2 - 2\Sigma BC . \cos\lambda, \text{ \&c.}$$

Making the above substitutions we get

$$\Delta_x = \tfrac{1}{2}\begin{vmatrix} y_1', & z_1', & 1 \\ \vdots & \vdots & \vdots \end{vmatrix} = A \begin{vmatrix} y_1, & z_1, & 1 \\ \vdots & \vdots & \vdots \end{vmatrix}$$

$$= A \begin{vmatrix} y_1, & z_1, & x_1+y_1+z_1+u_1 \\ \vdots & \vdots & \vdots \end{vmatrix} = A(y_1 z_2 u_3) + A(y_1 z_2 x_3).$$

Denoting the determinants

$$(y_1 z_2 u_3), \quad -(x_1 z_2 u_3), \quad (x_1 y_2 u_3), \quad -(x_1 y_2 z_3)$$

by X, Y, Z, U respectively, and substituting these results in (2), we get

$$(\Delta PQR)^2 = A^2(X-U)^2 + B^2(Y-U)^2 + C^2(Z-U)^2$$
$$-2BC(Y-U)(Z-U)\cos\lambda - 2CA(Z-U)(X-U)\cos\mu$$
$$-2AB(X-U)(Y-U)\cos\nu.$$

Now coefficient of $U^2 = \Sigma A^2 - 2\Sigma BC.\cos\lambda = D^2$ and coefficient of $-2AXU = A - B\cos\nu - C\cos\mu = D\cos\lambda'$. Hence

$$(\Delta PQR)^2 = \Sigma(AX)^2 - 2\Sigma BC.YZ\cos\lambda - 2\Sigma AD.XU\cos\lambda',$$

consisting of four, three, and three terms respectively.

We get the corresponding result in quadriplanars by the substitution of $x = A\alpha/3V$, &c., where V is the volume of the tetrahedron of reference. Thus, since

$$-U = (x_1 y_2 z_3) = (\alpha_1\beta_2\gamma_3)\,ABC/(3V)^3,$$

we have

$$(\Delta PQR)^2/\kappa^2 = \Sigma(\alpha_1\beta_2\gamma_3)^2 - 2\Sigma(\alpha_1\beta_2\gamma_3)(\beta_1\gamma_2\delta_3)\cos\lambda',$$

where κ denotes $ABCD/(3V)^3$.

4. To find the perpendicular from a point on a plane in homogeneous coordinates, we may adopt the above method of changing to oblique Cartesians, and substituting in the expression obtained for p in this system of coordinates. The required result is, however, more easily obtained by expressing the equivalent of α in ordinary rectangular Cartesians (corresponding to $x\cos\alpha + y\sin\alpha - p$ for a straight line), viz.,

$$\alpha = l_1 x + m_1 y + n_1 z - p_1,$$

where $l_1^2 + m_1^2 + n_1^2 = 1$, and substituting in the expression for the perpendicular on a plane in rectangular Cartesians; when we shall get for the perpendicular from the point $(\alpha', \beta', \gamma', \delta')$

on the plane $l\alpha + m\beta + n\gamma + r\delta = 0$ in quadriplanars

$$(l\alpha' + m\beta' + n\gamma' + r\delta') / (L^2 + M^2 + N^2)^{\frac{1}{2}},$$

where

$$L = ll_1 + ml_2 + nl_3 + rl_4, \quad M = lm_1 + mm_2 + nm_3 + rm_4,$$
$$N = ln_1 + mn_2 + nn_3 + rn_4.$$

Now, in the expression $L^2 + M^2 + N^2$, the coefficients of l^2, m^2, n^2, r^2 are each $= 1$, and the coefficient of $2lm = l_1l_2 + m_1m_2 + n_1n_2 =$ cosine of angle between the perpendiculars on the planes α and $\beta = -$ cosine of angle between the planes intersecting in $CD = -\cos\nu$; so that

$$L^2 + M^2 + N^2 = \Sigma l^2 - 2\Sigma mn\cos\lambda - 2\Sigma lr\cos\lambda',$$

consisting of four, three, and three terms respectively.

The perpendicular from the point (x', y', z', u') on the plane $lx + my + nz + ru = 0$ in tetrahedrals is readily deduced from this, and is equal to

$$\frac{3V(lx' + my' + nz' + ru')}{(\Sigma A^2l^2 - 2\Sigma BCmn\cos\lambda - 2\Sigma ADlr\cos\lambda')^{\frac{1}{2}}}.$$

Cor. The volume of a tetrahedron, in terms of the coordinates of its vertices, may be deduced from the foregoing results by writing down the equation to the plane through three of the vertices, when we shall get

$$3 \text{ vol.} = (\alpha_1\beta_2\gamma_3\delta_4)\, ABCD\,(3V)^3.$$

AN ELEMENTARY PROOF OF A PROPERTY OF GROUPS OF FINITE ORDER.

By *Prof. W. Burnside.*

In the last chapter of his *Calcul des Probabilités* M. Poincaré dealt with the problem of shuffling a pack of cards. His treatment of the problem involved the approximate evaluation, when m is a large number, of the expression

$$(p_1S_1 + p_2S_2 + \ldots + p_NS_N)^m,$$

where $S_1, S_2, \ldots, S_N$ are the distinct operations of a group of finite order N, and $p_1, p_2, \ldots, p_N$ are real positive (or zero) quantities, whose sum is unity; and he arrived at the evaluation by using the characteristic equation of the group.

The most fundamental property of a group of finite order is that when its distinct operations are multiplied by any one of them, each appears once in the system of products. I propose to show that the evaluation of the above expression follows directly from the property.

1. I consider first the simplest case, *viz.*, that in which $p_1, p_2, \ldots, p_N$ are real, non-zero, positive quantities, whose sum is unity. Since

$$S_j \sum_{i=1}^{i=N} S_i = \sum_{i=1}^{i=N} S_i$$

it follows that, whatever numbers $q_1, q_2, \ldots, q_N$ may be,

$$\sum_i q_i S_i \,.\, \sum_i S_i = \sum_i q_i \,.\, \sum_i S_i.$$

Further, writing

$$p_{ik} \equiv p_j, \quad \text{if } S_i S_k = S_j,$$

so that the N numbers p_{ik} $(i = 1, 2, \ldots, N)$ are the N numbers p_i in some altered sequence, then

$$\sum_i p_i S_i \,.\, S_k = \sum_j p_{jk^{-1}} S_j,$$

and therefore the sum of the coefficients in $(\sum_i p_i S_i)^m$ is $(\sum p_i)^m$ or unity. Hence, if we write

$$(\sum_i p_i S_i)^m = \frac{1}{N} \sum_i S_i + \sum_i f_i(m)\, S_i,$$

then

$$\sum_i f_i(m) = 0.$$

Now

$$\frac{1}{N} \sum_i S_i + \sum_k f_k(m+1)\, S_k = (\sum_i p_i S_i)^{m+1}$$

$$= \left(\frac{1}{N} \sum_i S_i + \sum_j f_i(m)\, S_i \right) \sum_j p_j S_j;$$

and

$$f_k(m+1) = \text{coefficient of } S_k \text{ in } \sum_{ij} f_i(m)\, p_j S_i S_j$$

$$= \sum_i f_i(m)\, p_{i^{-1}k}.$$

Since $\sum_i f_i(m) = 0$, some of the terms in the sum on the right are positive and some negative. Hence

$$|f_k(m+1)| < |\Sigma' f_i(m)\, p_{i^{-1}k}|,$$

where Σ' extends to the positive or to the negative terms of $\sum_i f_i(m)\, p_{i^{-1}k}$ according as the sum of the positive terms is numerically greater than the sum of the negative terms or *vice versâ*.

Denote by $F(m)$ the greatest of the quantities $|f_i(m)|$, $(i=1, 2, \ldots, N)$; and by p_μ the least of the numbers p_s. Then

$$|\Sigma' f_i(m)\, p_{i^{-1}k}| < (1-p_\mu)\, F(m),$$

so that
$$|f_k(m+1)| < (1-p_\mu)\, F(m),$$

and
$$F(m+1) < (1-p_\mu)\, F(m).$$

If p_ν' denotes the greatest of the quantites $\left|p_i - \frac{1}{N}\right|$, this last formula gives

$$F(m) < p_\nu'(1-p_\mu)^{m-1}.$$

Hence in the formula

$$(\Sigma p_i S_i)^m = \frac{1}{N}\Sigma S_i + \Sigma f_i(m)\, S_i,$$

$|f_i(m)|$ is less than $p_\nu'(1-p_\mu)^{m-1}$ for all values of i and m; and since $1-p_\mu$ is a proper fraction $(\Sigma p_i S_i)^m$ approaches the value $\frac{1}{N}\Sigma S_i$ as m increases.

2. Suppose next that some of the p's are zero, the rest being real and positive with unity for their sum. It is clear that if the operations with non-zero coefficients all belong to a sub-group, it is only operations of this sub-group which can appear in $(\sum_i p_i S_i)^m$. Denote then by $S_i (i=1, 2, \ldots, N)$ the distinct operations of the group which is generated by the operations with non-zero coefficients in $\Sigma p_i S_i$. If among these the identical operation occurs, there must be some smallest integer m_0, such that

$$(\sum_i p_i S_i)^{m_0} = \sum_i q_i S_i,$$

where no q is zero, while the sum of the q's is unity. In fact if S_1 is the identical operation and p_1 is not zero, a sufficiently higher power of $\sum_i p_i S_i$ contains every sequence of one, two,

three or more generating operations. In this case, with the previous notation,

$$(\Sigma_i p_i S_i)^{m_0 m} = \frac{1}{N} \Sigma_i S_i + \Sigma_i g_i(m)\, S_i,$$

where $$|g_i(m)| < q_\mu'(1 - q_\mu)^{m-1};$$

and therefore as M increases $(\Sigma_i p_i S_i)^M$ approaches the value $\frac{1}{N}\Sigma_i S_i$, as in the previous case. If however the identical operation does not occur in $\Sigma_i p_i S_i$ this is not necessarily true. In fact the operations with non-zero coefficients might be such that the product of any μ of them in any sequence belong to a self-conjugate sub-group. In this case it is obvious that $(\Sigma_i p_i S_i)^m$ would not approach $\frac{1}{N}\Sigma_i S_i$ as m increases.

3. If the sum of the p's is unity, but no supposition is made as to their all being positive, or even real, it is still true that

$$\Sigma_i p_i S_i = \frac{1}{N} \Sigma_i S_i + \Sigma_i q_i S_i,$$

where $$\Sigma q_i = 0.$$

Since $$\Sigma_i S_i \,.\, \Sigma_i q_i S_i = \Sigma_i q_i \,.\, \Sigma_i S_i = 0,$$

$$(\Sigma_i p_i S_i)^m = \frac{1}{N} \Sigma_i S_i + (\Sigma_i q_i S_i)^m.$$

In particular suppose that

$$\Sigma_i q_i S_i = \alpha (S_1 - S_2),$$

where S_1 is the identical operation and S_2 an operation of order two. Then by direct multiplication

$$(S_1 - S_2)^m = 2^{m-1}(S_1 - S_2),$$

and $$(\Sigma_i p_i S_i)^m = \frac{1}{N} \Sigma_i S_i + \tfrac{1}{2}(2\alpha)^m (S_1 - S_2).$$

The right-hand side does not approach the limiting value $\frac{1}{N}\Sigma_i S_i$ unless $|\alpha|$ is less than $\frac{1}{2}$. The result that has been proved when the p's are zero or positive is therefore not true without limitations in other cases.

4. The most general form in which the result can be stated, without limitation, is the following.

If $S_1, S_2, \ldots, S_r$ is a set of operations, including identity, which generate the group $S_1, S_2, \ldots, S_N$; and if $p_1, p_2, \ldots, p_r$ are real, positive, non-zero quantities whose sum is unity, then $(\sum_1^r p_i S_i)^m$ approaches the limit $\frac{1}{N}\sum_1^N S_i$ as m increases.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy.*

XXXIX.

Further examples of conditionally convergent infinite double integrals.

1. I HAVE had occasion recently to work out, in connection with a question submitted to me by Mr. C. G. Darwin, the conditions for the convergence of the integral

$$(1) \qquad \int_0^\infty \int_0^\infty \frac{\exp\{mi\sqrt{(ax^2+2hxy+by^2)}\}}{(ax^2+2hxy+by^2)^s}\,dx\,dy,$$

where m, a, b, $ab-h^2$, and s are positive. The results are curious, and the integrals form an interesting addition to the various non-absolutely convergent double integrals that I have considered in a number of former notes.

I shall suppose $s<1$. If $s\geq 1$ the integral diverges, owing to the infinity at the origin.*

2. I shall consider first that the conditions under which the integral (1) converges as a double integral, *i.e.*, under which

$$(2) \qquad \phi(H, K)=\int_0^H\int_0^K \frac{\exp\{mi\sqrt{(ax^2+2hxy+by^2)}\}}{(ax^2+2hxy+by^2)^s}\,dx\,dy$$

tends to a limit when H and K tend simultaneously to infinity. I begin by proving a formula which I shall appeal to frequently.

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* The integral converges absolutely at infinity if $s>1$ and non-absolutely if $s=1$.

Let

$$(3)\quad F(H,K)=\int_0^H\int_0^K f\{\sqrt{(ax^2+2hxy+by^2)}\}\,dx\,dy.$$

Then

$$(4)\quad F(H,K)=\int_0^K y\,dy\int_0^{H/y} f\{y\sqrt{(au^2+2hu+b)}\}\,du$$
$$=\psi(H,K)+\chi(H,K),$$

where

$$(5)\quad \psi(H,K)=\int_0^{H/K}du\int_0^K y\,f\{y\sqrt{(au^2+2hu+b)}\}\,dy,$$

and

$$(6)\quad \chi(H,K)=\int_{H/K}^{\infty}du\int_0^{H/u} y\,f\{y\sqrt{(au^2+2hu+b)}\}\,dy.$$

If in χ we make the substitutions $y=z/u=zv$, and then write y for z and u for v, we obtain

$$\chi(H,K)=\int_0^{K/H}du\int_0^H y\,f\{y\sqrt{(bu^2+2hu+a)}\}\,dy.$$

Hence

$$(7)\qquad F(H,K)=\psi(H,K)+\bar{\psi}(K,H),$$

where $\bar{\psi}$ is formed from ψ by interchanging a and b.

3. I suppose first that $s=0$. Applying the formulae (3)–(5) to the particular integral ϕ, and performing the integration with respect to y in (5), we obtain

$$(8)\qquad \psi(H,K)=\psi_1(H,K)+\psi_2(H,K),$$

where

$$\psi_1=\frac{K}{mi}\int_0^{H/K}\frac{\exp(miKU)}{U}\,du,$$

$$\psi_2=\frac{1}{m^2}\int_0^{H/K}\frac{\exp(miKU)-1}{U^2}\,du,$$

and

$$U=\sqrt{(au^2+2hu+b)}.$$

It is evident that ψ_2 is numerically less than a constant. I shall now prove that, if $h>0$, the same is true of ψ_1. If $h>0$, U is a steadily increasing function of u, and the real part of ψ_1 is

$$\frac{K}{m}\int_0^{H/K}\frac{\sin mKU}{U}\,du=\frac{K}{m}\int_{\sqrt{b}}^{\lambda}\frac{\sin mKU}{\sqrt{(aU^2-\Delta)}}\,dU$$
$$=\frac{K}{mh}\int_{\sqrt{b}}^{\mu}\sin mKU\,dU,$$

where $\Delta = ab - h^2$ and $\sqrt{b} < \mu < \lambda$. This is numerically less than a constant; and the same argument applies to the imaginary part of ψ_1. Thus ψ is numerically less than a constant; and the same holds of $\bar{\psi}$ and so of ϕ.

But it will easily be verified that, if

$$g(x, y) = (ax^2 + 2hxy + by^2)^{-s},$$

where a, h, b, and s are positive, then

$$\frac{\partial g}{\partial x} < 0, \quad \frac{\partial g}{\partial y} < 0, \quad \frac{\partial^2 g}{\partial x \partial y} > 0.$$

And from this it follows, by the obvious analogue for integrals of a theorem concerning the convergence of double series which I have proved elsewhere*, that the integral (1) is convergent for $s > 0$.

Thus, *when* $h > 0$, *the integral* (1) *is convergent as a double integral for* $s > 0$.

4. This conclusion no longer holds when $h \leq 0$. In the first place the range of integration with respect to u includes a value of u for which $aU^2 - \Delta = 0$; and in the second place the derivatives of $f(x, y)$ are no longer of constant sign. The condition for convergence is in fact $s > \frac{1}{4}$. The discussion is substantially the same when $h = 0$ and when $h < 0$, and I shall confine myself, for simplicity, to the first case. There is obviously no loss of generality in supposing $a = b = 1$. We have then

$$(9) \quad \psi(H, K) = \int_0^{HK} \frac{du}{(u^2+1)^s} \int_0^K \exp\{miy\sqrt{(u^2+1)}\}\, y^{1-2s}\, dy.$$

I suppose first that $\frac{1}{2} < s < 1$. Integrating by parts with respect to y, we obtain

$$(10) \qquad \psi(H, K) = \theta_1(H, K) + \theta_2(H, K),$$

where

$$\theta_1(H, K) = \frac{K^{1-2s}}{mi} \int_0^{HK} \frac{\exp\{miK\sqrt{(u^2+1)}\} - 1}{(u^2+1)^{s+\frac{1}{2}}}\, du,$$

$$\theta_2(H, K) = -\frac{1-2s}{mi} \times \int_0^{HK} \frac{du}{(u^2+1)^{s+\frac{1}{2}}} \int_0^K \left[\exp\{miy\sqrt{(u^2+1)}\} - 1\right] y^{-2s}\, dy.$$

* *Proc. London Math. Soc.*, ser. 2, vol. i., p. 124.

It is evident that $\theta_1(H, K)$ and $\theta_1(K, H)$ tend to zero; and, in virtue of (7),

$$\theta_2(H, K)+\theta_2(K, H) = -\frac{1-2s}{mi}\int_0^H\int_0^K \frac{\exp\{mi\sqrt{(x^2+y^2)}\}-1}{(x^2+y^2)^{s+\frac{1}{2}}}\,dx\,dy.$$

The integral on the right-hand side converges absolutely when H and K tend to infinity; hence $\psi(H, K)$ tends to a limit, and so also does $\phi(H, K)$.

5. Next, suppose $\frac{1}{4}<s<\frac{1}{2}$. In this case

$$\psi(H, K)=\theta_1'(H, K)+\theta_2'(H, K), \tag{11}$$

where

$$\theta_1'(H, K)=\frac{K^{1-2s}}{mi}\int_0^{H/K}\frac{\exp\{miK\sqrt{(u^2+1)}\}}{(u^2+1)^{s+\frac{1}{2}}}\,du,$$

$$\theta_2'(H, K)=-\frac{1-2s}{mi} \times\int_0^{H/K}\frac{du}{(u^2+1)^{s+\frac{1}{2}}}\int_0^K \exp\{miy\sqrt{(u^2+1)}\}\,y^{-2s}\,dy.$$

Also, by (7),

$$\theta_2'(H, K)+\theta_2'(K, H) = -\frac{1-2s}{mi}\int_0^H\int_0^K\frac{\exp\{mi\sqrt{(x^2+y^2)}\}}{(x^2+y^2)^{s+\frac{1}{2}}}\,dx\,dy.$$

The integral on the right-hand side converges, when H and K tend to infinity, in virtue of what was proved in § 4; so that we need only discuss θ_1'. But

$$\theta_1'(H, K)=\frac{K^{1-2s}}{mi}\int_1^\lambda \frac{\exp(miKU)}{\sqrt{(U^2-1)}}\,U^{-2s}\,dU,$$

where $U=\sqrt{(u^2+1)}$ and $\lambda>1$. Dividing the integral into its real and imaginary parts, applying the second mean value theorem, and putting $U-1=t$, we reduce θ_1' to a sum of terms each of which is of the form

$$O(K^{1-2s})\int_1^\mu \begin{matrix}\cos\\ \sin\end{matrix}(mKt)\frac{dt}{\sqrt{t}}=O(K^{\frac{1}{2}-2s}).$$

These terms all tend to zero if $s>\frac{1}{4}$.

It follows that, *when* $h=0$, *the integral* (1) *is convergent as a double integral for* $s>\frac{1}{4}$. A similar argument applies when $h<0$.

6. I shall now consider the question of the convergence of the integral (1) as a *repeated* integral.

Let us write

$$(12)\quad f(y)=\int_0^\infty \frac{\exp\{mi\sqrt{(ax^2+2hxy+by^2)}\}}{(ax^2+2hxy+by^2)^s}\,dx.$$

This integral is certainly convergent whenever $s>0$. It follows that the repeated integral

$$(13)\qquad \int_0^\infty f(y)\,dy$$

is convergent whenever the double integral is convergent, that is to say for $s>0$ (if $h>0$) or for $s>\frac{1}{4}$ (if $h\leq 0$). The same remark applies to the other repeated integral; and the values of the three integrals are, in these circumstances, identical.

I shall now show, by a direct investigation, that the conditions just stated are necessary as well as sufficient for the convergence of the integral (13). It will then follow that the same is true of the double integral considered in §§ 2–5.

By means of the transformation

$$ax+hy=\Delta yu$$

we reduce $f(y)$ to the form

$$(14)\qquad Cy^{1-2s}\int_{h/\Delta}^\infty \frac{e^{Aiy\sqrt{(u^2+1)}}}{(u^2+1)^s}\,du,$$

where $A=\Delta/\sqrt{a}$ and C is a constant.

Suppose first $h>0$. The substitution $\sqrt{(u^2+1)}=t$ reduces the integral which occurs in (12) to the form

$$\int_\lambda^\infty e^{Aiyt}\frac{t^{1-2s}\,dt}{\sqrt{(t^2-1)}},$$

where $\lambda=\sqrt{(ab)}/\Delta>1$. This integral may be transformed by Cauchy's theorem into one taken along the line $t=\lambda+i\eta$ in the plane of the complex variable t, which is equal to

$$(15)\qquad ie^{iy\sqrt{b}}\int_0^\infty e^{-Ay\eta}\frac{(\lambda+i\eta)^{1-2s}\,d\eta}{\sqrt{(\lambda^2-1+2i\lambda\eta-\eta^2)}};$$

and it is easy to show that the last integral has an asymptotic expansion in powers of $1/y$, which begins with a term in $1/y$.

It follows that $f(y)$ is of the form

$$Dy^{-2s}e^{iy\sqrt{b}} + O(y^{-1-2s}),$$

when y is large, and consequently that the repeated integral is convergent if, and only if, $s > 0$.

This analysis has to be modified when $h \leq 0$, owing to the fact that the range $(h/\Delta, \infty)$ includes the point $u = 0$ for which $t = 1$. It will be enough to consider the case when $h = 0$, $\lambda = 1$. The integral (15) is then replaced by one whose asymptotic expansion begins with a term in $1/\sqrt{y}$, and $f(y)$ is of the form

$$Ey^{\frac{1}{2}-2s}e^{iy\sqrt{b}} + Fy^{-\frac{1}{2}-2s}e^{iy\sqrt{b}} + O(y^{-\frac{3}{2}-2s}).$$

Thus the repeated integral is convergent if, and only if, $s > \frac{1}{4}$.

7. We have thus arrived at the complete solution of our problem. *The necessary and sufficient condition for the convergence of the double integral* (1), *or of either of the associated repeated integrals, is* $s > 0$ *or* $s > \frac{1}{4}$, *according as* $h > 0$ *or* $h \leq 0$.

It is also interesting to consider the question of the convergence of the integral (1) regarded as the limit of an integral over the quadrant

$$0 \leq x, \ 0 \leq y, \ x^2 + y^2 \leq R^2.$$

I find that the necessary and sufficient condition for convergence is $s > \frac{1}{4}$, except in the particular case when $a = b$ and $h = 0$. In this particular case the condition is $s > \frac{1}{2}$. That this case should be exceptional is plainly not unnatural.

8. Integrals of the type considered in this note may sometimes be evaluated in finite terms. Thus, for example,

$$\int_0^\infty \int_0^\infty \frac{\cos m\sqrt{(x^2+y^2)}}{\sqrt{(x^2+y^2)}}\,dx\,dy = 0,$$

$$\int_0^\infty \int_0^\infty \frac{\sin m\sqrt{(x^2+y^2)}}{\sqrt{(x^2+y^2)}}\,dx\,dy = \frac{\pi}{2m},$$

$$\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{\sin m\sqrt{(ax^2+2hxy+by^2)}}{\sqrt{(ax^2+2bxy+by^2)}}\,dx\,dy = \frac{\pi}{2m\sqrt{(ab-h^2)}}.$$

It should be observed that these integrals cannot be calculated in the obvious manner by transformation to polar coordinates, on account of the nature of their convergence.

Integrals of the more general type

$$\int_0^\infty \int_0^\infty \frac{\exp\{mi\sqrt{(ax^2+2hxy+by^2)}+pix+qiy\}}{(ax^2+2hxy+by^2)^s}\,dx\,dy$$

also give rise to interesting problems of convergence. But more interesting than the integrals are the analogous series, such as

$$\Sigma\Sigma \frac{\sin\alpha\sqrt{(m^2+n^2)}}{\sqrt{(m^2+n^2)}}.$$

This series is convergent either as a double or as a repeated series. If $0<\alpha<2\pi$, and m and n range from 0 to ∞, the terms for which one of m or n is zero being divided by 2, and the term for which both are zero by 4, this last term moreover being assigned the value α, then the sum of the series is $\pi/2\alpha$. I shall return to these series on another occasion.

Addendum to a previous paper.

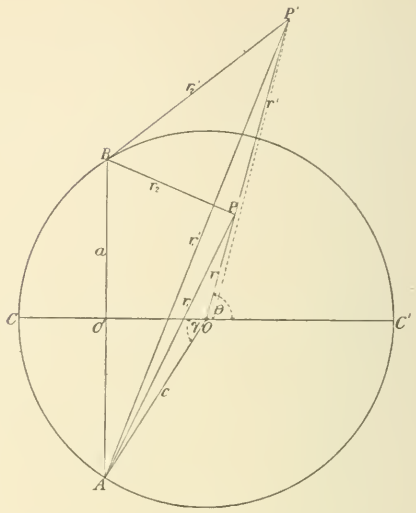
There is a slight obscurity in the proof of Theorem C of the paper "Some theorems concerning Dirichlet's series," by G. H. Hardy and J. E. Littlewood (*Messenger*, vol. 43, p. 141, top). The argument obviously depends on the assumption that $r>-p/(p+1)$, which is not stated explicitly.

ON THE POTENTIAL AND FORCE FUNCTION OF AN ELECTRIFIED SPHERICAL BOWL.

By *J. R. Wilton, M.A., D.Sc.*, Assistant Lecturer in Mathematics at the University of Sheffield.

1. I am indebted to Sir George Greenhill for the suggestion that I should undertake the work of the following short paper, for much care and trouble which he gave to explaining to me exactly what progress had hitherto been made with the problem, and for the interest he has taken in the progress of the work.

In the figure let P be the point at which the potential is required, let OC be the axis of the bowl, O being the centre of the sphere of which it forms a part. Let ACB be the section of the bowl by the plane of the paper; and let P' be the point inverse to P with regard to the sphere.



We shall use the following notation: x and y are the Cartesian coordinates of P with regard to the axes $O'O$, $O'B$;

$$OP=r,\quad OP'=r',\quad AP=r_1,\quad BP=r_2,\quad AP'=r_1',\quad BP'=r_2';$$

$$AO=c,\quad O'B=O'A=a=c\sin\gamma;$$

and we define the variables ω and ω' by the equations

$$r_1+r_2=2a\operatorname{cosec}\omega,$$

$$r_1'+r_2'=2a\operatorname{cosec}\omega'.$$

We have

$$\frac{r_1'}{r_1}=\frac{r_2'}{r_2}=\frac{c}{r}=\frac{r'}{c}=\frac{r_1'+r_2'}{r_1+r_2}=\frac{\sin\omega}{\sin\omega'}.$$

The potential at P, due to the freely electrified bowl, is

$$\phi=\frac{V}{\pi}\left\{\sin^{-1}\frac{2a}{r_1+r_2}+\frac{c}{r}\sin^{-1}\frac{2a}{r_1'+r_2'}\right\},$$

where V is the potential on the bowl itself. In terms of ω and ω'

$$\phi=\frac{V}{\pi}\left(\omega+\omega'\frac{\sin\omega}{\sin\omega'}\right).$$

Now ω is itself a potential function at P, and the force function (or Stokes' function), corresponding to

$$\phi=\frac{V}{\pi}\omega,$$

is $$M = V\sqrt{\{4a^2 - (r_1 - r_2)^2\}},$$

where we define M by the equations,

$$\frac{\partial\phi}{\partial x} = \frac{1}{2\pi y}\frac{\partial M}{\partial y}, \qquad \frac{\partial\phi}{\partial y} = -\frac{1}{2\pi y}\frac{\partial M}{\partial x}.$$

On the other hand, ω' is a potential at P', and therefore* $(c/r)\omega' = \omega' \sin\omega/\sin\omega'$ is a potential at P. It follows that the Stokes' function at P', corresponding to $\phi = V\omega'/\pi$, is

$$M' = V\sqrt{\{4a^2 - (r_1' - r_2')^2\}};$$

and Sir George Greenhill notes (though I believe that this remark has not been published) that corresponding to M' at P' there is a Stokes' function†

$$(r/c)M' = V\sqrt{\{4r^2\sin^2\gamma - (r_1 - r_2)^2\}}$$

at P.

The question naturally arises: What is the Stokes' function corresponding to

$$\phi = V\omega'\sin\omega/\pi\sin\omega'?$$

and what is the potential function corresponding to

$$M = V\sqrt{\{4r^2\sin^2\gamma - (r_1 - r_2)^2\}}?$$

2. The reply to these two questions can very easily be given if we take the coordinates ω and ω' as independent variables.

We readily find

$$x^2 + y^2\cos^2\omega = a^2\cot^2\omega,$$

$$x = c\operatorname{cosec}\omega\cot\omega(\cos\omega' - \cos\gamma\cos\omega),$$

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* W. D. Niven, *Proc. L. M. S.* vol. viii., p. 66; also W. Burnside, *id.* vol. xxv., p. 99.

† Sir George Greenhill remarks that the four Stokes' functions,

$$X = \sqrt{\{(r_1 + r_2)^2 - 4a^2\}}, \qquad Y = \sqrt{\{4a^2 - (r_1 - r_2)^2\}},$$
$$Z = \sqrt{\{4r^2\sin^2\gamma - (r_1 - r_2)^2\}}, \quad W = \sqrt{\{(r_1 + r_2)^2 - 4r^2\sin^2\gamma\}},$$

are connected by two simple relations, namely,

$$X = Z\sin\gamma + W\cos\gamma,$$
$$Y = -Z\cos\gamma + W\sin\gamma.$$

When X, Y, Z and W are expressed in terms of ω and ω' these relations follow immediately. For

$$X = 2a\cot\omega,$$
$$Y = 2c(\cos\omega' - \cos\gamma\cos\omega)/\sin\omega,$$
$$Z = 2c(\cos\omega - \cos\gamma\cos\omega')/\sin\omega,$$
$$W = 2a\cos\omega'/\sin\omega.$$

A geometrical interpretation can be given for each of the four quantities X, Y, Z, W, but it is not sufficiently simple to be of much interest.

and therefore

$$y^2 = c^2 \operatorname{cosec}^2\omega \{\sin^2\gamma - \operatorname{cosec}^2\omega\,(\cos\omega' - \cos\gamma\cos\omega)^2\}.$$

Whence, if

$$D \equiv c^4(\cos^2\omega - 2\cos\gamma\cos\omega\cos\omega' + \cos^2\omega'),$$

we find

$$cyJ \equiv cy\left(\frac{\partial x}{\partial\omega'}\frac{\partial y}{\partial\omega} - \frac{\partial x}{\partial\omega}\frac{\partial y}{\partial\omega'}\right) = D\sin\omega'\operatorname{cosec}^5\omega,$$

$$y^2\left[\left(\frac{\partial x}{\partial\omega}\right)^2 + \left(\frac{\partial y}{\partial\omega}\right)^2\right] = D\sin^2\omega'\operatorname{cosec}^6\omega$$
$$= y^2\left[\left(\frac{\partial x}{\partial\omega'}\right)^2 + \left(\frac{\partial y}{\partial\omega'}\right)^2\right],$$

$$y^2\left[\frac{\partial x}{\partial\omega}\frac{\partial x}{\partial\omega'} + \frac{\partial y}{\partial\omega}\frac{\partial y}{\partial\omega'}\right] = D\sin\omega'\operatorname{cosec}^7\omega\,(\cos\omega\cos\omega' - \cos\gamma).$$

But, from the relations between M and ϕ, we have at once

$$\frac{\partial\phi}{\partial\omega} = \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial\omega} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial\omega}, \qquad \frac{1}{2\pi y}\frac{\partial M}{\partial\omega} = \frac{\partial\phi}{\partial x}\frac{\partial y}{\partial\omega} - \frac{\partial\phi}{\partial y}\frac{\partial x}{\partial\omega},$$

$$\frac{\partial\phi}{\partial\omega'} = \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial\omega'} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial\omega'}, \qquad \frac{1}{2\pi y}\frac{\partial M}{\partial\omega'} = \frac{\partial\phi}{\partial x}\frac{\partial y}{\partial\omega'} - \frac{\partial\phi}{\partial y}\frac{\partial x}{\partial\omega'};$$

whence

$$\frac{\partial(\phi, x)}{\partial(\omega, \omega')} = J\frac{\partial\phi}{\partial y}, \qquad \frac{\partial(\phi, y)}{\partial(\omega, \omega')} = -J\frac{\partial\phi}{\partial x},$$

$$\frac{\partial M}{\partial\omega} = \frac{2\pi y}{J}\left[\frac{\partial\phi}{\partial\omega'}\left\{\left(\frac{\partial x}{\partial\omega}\right)^2 + \left(\frac{\partial y}{\partial\omega}\right)^2\right\} - \frac{\partial\phi}{\partial\omega}\left(\frac{\partial x}{\partial\omega}\frac{\partial x}{\partial\omega'} + \frac{\partial y}{\partial\omega}\frac{\partial y}{\partial\omega'}\right)\right],$$

$$\frac{\partial M}{\partial\omega'} = \frac{2\pi y}{J}\left[\frac{\partial\phi}{\partial\omega'}\left(\frac{\partial x}{\partial\omega}\frac{\partial x}{\partial\omega'} + \frac{\partial y}{\partial\omega}\frac{\partial y}{\partial\omega'}\right) - \frac{\partial\phi}{\partial\omega}\left\{\left(\frac{\partial x}{\partial\omega'}\right)^2 + \left(\frac{\partial y}{\partial\omega'}\right)^2\right\}\right],$$

From these equations (which evidently hold for any transformation of coordinates) we plainly have

$$\frac{\partial M}{\partial\omega} = 2\pi c\left\{\frac{\sin\omega'}{\sin\omega}\frac{\partial\phi}{\partial\omega'} - \frac{\cos\omega\cos\omega' - \cos\gamma}{\sin^2\omega}\frac{\partial\phi}{\partial\omega}\right\},$$

$$\frac{\partial M}{\partial\omega'} = 2\pi c\left\{\frac{\cos\omega\cos\omega' - \cos\gamma}{\sin^2\omega}\frac{\partial\phi}{\partial\omega'} - \frac{\sin\omega'}{\sin\omega}\frac{\partial\phi}{\partial\omega}\right\}.$$

From the figure we see that

$$\cos\theta = \frac{x - c\cos\gamma}{r} = \frac{\cos\omega\cos\omega' - \cos\gamma}{\sin\omega\sin\omega'};$$

hence the simplest form in which the relations between M and ϕ can be expressed is

$$\frac{\partial M}{\partial \omega} = 2\pi r \left(\frac{\partial \phi}{\partial \omega'} - \cos\theta \frac{\partial \phi}{\partial \omega} \right),$$

$$\frac{\partial M}{\partial \omega'} = 2\pi r \left(\cos\theta \frac{\partial \phi}{\partial \omega'} - \frac{\partial \phi}{\partial \omega} \right).$$

The equation satisfied by ϕ follows immediately: it is

$$\frac{\partial^2 \phi}{\partial \omega^2} - 2\cos\theta \frac{\partial^2 \phi}{\partial \omega \partial \omega'} + \frac{\partial^2 \phi}{\partial \omega'^2} + 2 \frac{\cos\omega' - \cos\gamma\cos\omega}{\sin^2\omega \sin\omega'} \frac{\partial \phi}{\partial \omega'} = 0.$$

The most general solution independent of γ is

$$\phi = A\omega + B + C(\omega' + \alpha) \sin\omega / \sin\omega',$$

which includes the special case

$$\phi = \frac{V}{\pi} (\omega + \omega' \sin\omega / \sin\omega')$$

for the potential of the bowl.

It follows from the definitions that $\sin\omega$ and $\sin\omega'$ are always positive, and as will be seen presently we may suppose both ω and ω' to lie between 0 and π.

On the sphere we have $r = c$, and therefore $\sin\omega = \sin\omega'$, so that either

$$\omega = \omega' \text{ or } \omega + \omega' = \pi.$$

Now
$$\cos\theta = \frac{\cos\omega\cos\omega' - \cos\gamma}{\sin\omega \sin\omega'}.$$

Consequently, (1) when $\omega = \omega'$,

$$\cos\theta = \frac{\cos^2\omega - \cos\gamma}{\sin^2\omega} = -\frac{\cos\gamma - \cos^2\omega}{1 - \cos^2\omega},$$

so that $\cos\theta$ is, if negative, numerically less than $\cos\gamma$, *i.e.*, $\theta < \pi - \gamma$, while ω varies from $\frac{1}{2}\gamma$ to $\frac{1}{2}\pi$; (2) when $\omega + \omega' = \pi$,

$$\cos\theta = -\frac{\cos\gamma + \cos^2\omega}{\sin^2\omega}$$

is negative and numerically greater than $\cos\gamma$, *i.e.*, $\theta > \pi - \gamma$, while ω varies from $\frac{1}{2}\pi$ to $\frac{1}{2}\pi + \frac{1}{2}\gamma$.

It has been assumed that $\gamma < \frac{1}{2}\pi$, but it is easy to see that the same results as to the value of θ hold when $\cos\gamma$ is negative.

Thus $\omega + \omega' = \pi$ is the equation of the bowl, and $\omega = \omega'$ is the equation of the remainder of the sphere.

When P is within the sphere ω varies from its least value $\frac{1}{2}\gamma$ at C' to its maximum value $\frac{1}{2}\pi+\frac{1}{2}\gamma$ at C, passing through $\frac{1}{2}\pi$ as P crosses AB; while ω' varies from 0 when P is at the centre of the sphere to $\frac{1}{2}\pi$, when P is at A or B. On the other hand, when P is outside the sphere, ω varies from 0 at infinity to $\frac{1}{2}\pi$ at A or B, while ω' varies from $\frac{1}{2}\gamma$ to $\frac{1}{2}\pi+\frac{1}{2}\gamma$, passing through $\frac{1}{2}\pi$ as P crosses the bowl which is the inverse of the disc AB. Hence the extreme limits for both ω and ω' are 0 to $\frac{1}{2}\pi+\frac{1}{2}\gamma$; and, as γ is less than π, neither ω nor ω' is ever so great as π. They are both discontinuous in crossing the bowl ACB.

It follows that at the origin we must have

$$\omega=\gamma, \quad \omega'=0,$$

and at infinity $\omega=0, \quad \omega'=\gamma.$

Therefore the solution

$$\phi=\frac{V}{\pi}(\omega+\omega'\sin\omega/\sin\omega')$$

vanishes at infinity, is constant and equal to V over the bowl $\omega+\omega'=\pi$, and it never becomes infinite. Hence it is, as is well known, the potential function of the freely electrified bowl.*

On the remainder of the spherical surface ϕ is not constant, but is equal to $2V\omega/\pi$, and varies from V at A or B to $V\gamma/\pi$ at C'.

Another solution of the equation for ϕ is

$$\phi=\frac{E}{c\pi}\left\{(\pi-\omega')\frac{\sin\omega}{\sin\omega'}-\omega\right\};$$

and this evidently vanishes at infinity and over the bowl $\omega+\omega'=\pi$, while in the neighbourhood of the centre of the sphere it becomes infinite like

$$\frac{E}{c\pi}\pi\frac{\sin\omega}{\sin\omega'}=\frac{E}{r},$$

---

* The notation ω, ω' was chosen in order to emphasize the analogy between the potential of the freely electrified bowl and the gravitational potential of a material bowl, namely,

$$U=c\sigma(\Omega+\Omega' c/r),$$

where

σ is the superficial density in gravitational units,

Ω is the apparent area of the disc AB at P,

Ω' is the apparent area of the disc AB at P'.

(G. Greenhill, *Amer. Jour. of Math.*, vol. xxxiii., No 4, p. 386.)

i.e., ϕ is the potential function due to the uninsulated bowl in presence of a point charge E at the centre O.

From the formulæ at the beginning of this paragraph we readily find that the resultant electric intensity at any point is R, where

$$R^2 = \frac{c^2 \sin^4 \omega}{D} \left\{ \left(\frac{\partial \phi}{\partial \omega} \right)^2 - 2 \cos\theta \, \frac{\partial \phi}{\partial \omega} \frac{\partial \phi}{\partial \omega'} + \left(\frac{\partial \phi}{\partial \omega'} \right)^2 \right\}.$$

And in the case of the freely electrified bowl

$$\begin{aligned} R^2 = \left(\frac{Vc}{\pi} \right)^2 \frac{\sin^4 \omega}{D \sin^4 \omega'} \big[& \sin^2 \omega' \{ 2 (1 + \cos\gamma) - (\cos\omega + \cos\omega')^2 \} \\ & + 2 (1 + \cos\gamma) \, \omega' \sin\omega' (\cos\omega - \cos\omega') \\ & + \omega'^2 (\cos^2 \omega - 2 \cos\gamma \cos\omega \cos\omega' + \cos^2 \omega') \big], \end{aligned}$$

so that on the bowl itself, where $\omega + \omega' = \pi$,

$$R = \pm \frac{V}{\pi c} (\tan\omega' - \omega').$$

On the inside (the concave side) of the bowl ω' is less than $\frac{1}{2}\pi$, varying from $\frac{1}{2}\pi - \frac{1}{2}\gamma$ at C to $\frac{1}{2}\pi$ at A or B, while on the outside ω' is greater than $\frac{1}{2}\pi$. Denoting ω' on the inside of the bowl by ω_i', and on the outside by ω_0', we have

$$\omega_0' + \omega_i' = \pi.$$

Thus, if σ is the density,

$$\begin{aligned} \sigma_i &= \frac{R_i}{4\pi} = \frac{V}{4\pi^2 c} (\tan\omega_i' - \omega_i'), \\ \sigma_0 &= \frac{R_0}{4\pi} = \frac{V}{4\pi^2 c} (\omega_0' - \tan\omega_0') \\ &= \frac{V}{4\pi^2 c} (\pi + \tan\omega_i' - \omega_i') \\ &= \sigma_i + \frac{V}{4\pi c}. \end{aligned}$$

These results are in agreement with Lord Kelvin's well-known formulæ.

3. We proceed to determine the Stokes' functions corresponding to the two values of ϕ given in the last paragraph.

In the first place, it is evident that, corresponding to the term $\sin\omega / \sin\omega' = c/r$, we must have a term

$$2\pi c \cos\theta = 2\pi c \, \frac{\cos\omega \cos\omega' - \cos\gamma}{\sin\omega \sin\omega'}.$$

This is very easily verified, for, putting $\phi = \sin\omega/\sin\omega'$ in the equations connecting M and ϕ, we find

$$\frac{\partial M}{\partial \omega} = -2\pi c \left\{ \frac{\sin\omega'}{\sin\omega} \frac{\sin\omega\cos\omega'}{\sin^2\omega'} + \frac{\cos\omega\cos\omega' - \cos\gamma}{\sin^2\omega} \frac{\cos\omega}{\sin\omega'} \right\}$$

$$= -2\pi c \frac{\cos\omega' - \cos\gamma\cos\omega}{\sin^2\omega\sin\omega'},$$

$$\frac{\partial M}{\partial \omega'} = -2\pi c \frac{\cos\omega - \cos\gamma\cos\omega'}{\sin\omega\sin^2\omega'}.$$

Whence $$M = 2\pi c \frac{\cos\omega\cos\omega' - \cos\gamma}{\sin\omega\sin\omega'} = 2\pi c\cos\theta.$$

Corresponding to $\phi = \omega$ we find

$$\frac{\partial M}{\partial \omega} = -2\pi c \frac{\cos\omega\cos\omega' - \cos\gamma}{\sin^2\omega},$$

$$\frac{\partial M}{\partial \omega'} = -2\pi c \frac{\sin\omega'}{\sin\omega},$$

and therefore $$M = 2\pi c \frac{\cos\omega' - \cos\gamma\cos\omega}{\sin\omega}.$$

And, finally, corresponding to $\phi = \omega'\sin\omega/\sin\omega'$, we find

$$\frac{\partial M}{\partial \omega} = 2\pi c \left\{ 1 - \frac{\omega'(\cos\omega' - \cos\gamma\cos\omega)}{\sin^2\omega\sin\omega'} \right\},$$

$$\frac{\partial M}{\partial \omega'} = 2\pi c \left\{ \frac{\cos\omega\cos\omega' - \cos\gamma}{\sin\omega\sin\omega'} - \omega' \frac{\cos\omega - \cos\gamma\cos\omega'}{\sin\omega\sin^2\omega'} \right\}.$$

Whence

$$M = 2\pi c \left(\omega + \omega' \frac{\cos\omega\cos\omega' - \cos\gamma}{\sin\omega\sin\omega'} \right) = 2\pi c\,(\omega + \omega'\cos\theta).$$

Thus corresponding to the potential of the freely electrified bowl, namely,

$$\phi = \frac{V}{\pi}(\omega + \omega'\sin\omega/\sin\omega'),$$

we have

$$M = 2Vc \left\{ \frac{\cos\omega' - \cos\gamma\cos\omega}{\sin\omega} + \omega + \omega' \frac{\cos\omega\cos\omega' - \cos\gamma}{\sin\omega\sin\omega'} \right\}$$

$$= V\left[\sqrt{\{4a^2 - (r_1 - r_2)^2\}} + 2c\sin^{-1}\frac{2a}{r_1 + r_2} + 2c\cos\theta\sin^{-1}\frac{2r\sin\gamma}{r_1 + r_2} \right]$$

in a more familiar notation.

Corresponding to the potential due to the uninsulated bowl in presence of the charge E at O, namely,

$$\phi = \frac{E}{c\pi}\left\{(\pi - \omega')\frac{\sin\omega}{\sin\omega'} - \omega\right\},$$

we have

$$M = 2E\left\{(\pi - \omega')\frac{\cos\omega\cos\omega' - \cos\gamma}{\sin\omega\sin\omega'} - \omega - \frac{\cos\omega' - \cos\gamma\cos\omega}{\sin\omega}\right\},$$

which may be interpreted in the same way.

Having obtained the value of M, we may immediately write down the capacity. In particular, the capacity of the freely electrified bowl is

$$\{M(c_i) - M(c_0)\}/4\pi V,$$

where the symbols $M(c_i)$ and $M(c_0)$ denote the values of M at C on the concave and on the convex side of the bowl respectively. This makes the capacity equal to

$$\frac{c}{\pi}(\gamma + \sin\gamma),$$

a known result.

Sir George Greenhill points out that this is the capacity of a sphere of the same *girth* as the bowl.

The value of ϕ for the bowl at potential $V + E/c$ under the influence of the point charge E at O is evidently

$$\phi = \left(\frac{E}{c} + \frac{V\omega'}{\pi}\right)\frac{\sin\omega}{\sin\omega'} + \frac{V\omega}{\pi},$$

corresponding to which we find the capacity

$$\frac{Vc}{E + Vc}\,\frac{c}{\pi}(\gamma + \sin\gamma),$$

which, as it should, becomes infinite for the uninsulated bowl, for which $E = -Vc$.

4. It is of some interest to determine the value of ϕ corresponding to

$$M = \pi\sqrt{\{4r^2\sin^2\gamma - (r_1 - r_2)^2\}}$$
$$= -2\pi c\,\mathrm{cosec}\,\omega\,(\cos\omega - \cos\gamma\cos\omega').$$

Substituting in the equations connecting M and ϕ we readily find

$$\frac{\partial\phi}{\partial\omega'} = -\frac{\sin^2\gamma\sin\omega\sin\omega'}{(\cos\omega' - \cos\gamma\cos\omega)^2 - \sin^2\gamma\sin^2\omega},$$

$$\frac{\partial\phi}{\partial\omega} = \frac{\cos\gamma(\cos^2\omega + \cos^2\omega') - (1 + \cos^2\gamma)\cos\omega\cos\omega'}{(\cos\omega' - \cos\gamma\cos\omega)^2 - \sin^2\gamma\sin^2\omega}.$$

Whence

$$\phi = \tfrac{1}{2}\sin\gamma \log \frac{\cos(\omega-\gamma)-\cos\omega'}{\cos\omega'-\cos(\omega+\gamma)} + \omega\cos\gamma.$$

Since ω is itself a potential function we obtain a value of ϕ in the form

$$\phi = \tfrac{1}{2}\log \frac{\cos(\omega-\gamma)-\cos\omega'}{\cos\omega'-\cos(\omega+\gamma)}$$

$$= \pm \cosh^{-1}\frac{2a}{r_1 \sim r_2},$$

as is easily verified.

This value of ϕ has a logarithmic infinity at every point on the axis of the bowl. For, if either the numerator or the denominator of the logarithm is zero, $y=0$. If the numerator is zero

$$\cos\omega' = \cos(\omega-\gamma)$$

and $\quad x = c\,\mathrm{cosec}\,\omega\cot\omega\,(\cos\omega'-\cos\gamma\cos\omega) = a\cot\omega,$

varies from $-a\tan\frac{1}{2}\gamma$ to $+\infty$; but if the denominator is zero

$$x = -a\cot\omega,$$

and varies from $-\infty$ to $-a\tan\frac{1}{2}\gamma$; for in this case $\omega' = \omega+\gamma$, and therefore $\omega \not> \frac{1}{2}\pi - \frac{1}{2}\gamma$.

It is easy to form a value of M which vanishes over the bowl $\omega+\omega' = \pi$, namely,

$$M = -2\pi c\left\{\frac{\cos\omega-\cos\gamma\cos\omega'}{\sin\omega} + \frac{\cos\omega'-\cos\gamma\cos\omega}{\sin\omega}\right\}$$

$$= -2\pi c\,(1-\cos\gamma)\frac{\cos\omega+\cos\omega'}{\sin\omega}.$$

The corresponding value of ϕ is

$$\phi = \tfrac{1}{2}\sin\gamma\log\frac{\cos(\omega-\gamma)-\cos\omega'}{\cos\omega'-\cos(\omega+\gamma)} - \omega\,(1-\cos\gamma)$$

$$= \pm\sin\gamma\cosh^{-1}\frac{2a}{r_1\sim r_2} - (1-\cos\gamma)\sin^{-1}\frac{2a}{r_1+r_2}.$$

In this case M and ϕ are respectively the stream function and velocity potential, in the analogous hydrodynamical question, for the bowl in the presence of a semi-infinite line source in the axis, extending from C to the right, and a semi-infinite line sink of equal strength extending, in the axis, from C to the left.

We may obtain another form of the solution of the same problem by noticing that, since both

$$\frac{\cos\omega - \cos\gamma\cos\omega'}{\sin\omega} \quad \text{and} \quad \frac{\cos\omega' - \cos\gamma\cos\omega}{\sin\omega}$$

are possible values of M, it follows that

$$\cot\omega \quad \text{and} \quad \cos\omega'/\sin\omega$$

are possible values of M. Corresponding to

$$M = -2\pi a\cos\omega'/\sin\omega,$$

which is zero over the bowl $\omega' = \frac{1}{2}\pi$, we have

$$\phi = \tfrac{1}{2}\cos\gamma\log\frac{\cos(\omega-\gamma)-\cos\omega'}{\cos\omega'-\cos(\omega+\gamma)} - \omega\sin\gamma.$$

As they stand both ϕ and M are discontinuous at the bowl $\omega + \omega' = \pi$, but this disadvantage is easily removed by transforming to the more familiar coordinates r_1, r_2, and r. We have

$$M = -\pi\sqrt{\{(r_1+r_2)^2 - 4r^2\sin^2\gamma\}},$$

$$\phi = \cos\gamma\cosh^{-1}\frac{2a}{r_1\sim r_2} + \sin\gamma\cos^{-1}\frac{2a}{r_1+r_2},$$

where we have, for greater symmetry, added a constant $\frac{1}{2}\pi\sin\gamma$ to the value of ϕ. M and ϕ are now the stream function and velocity potential for the bowl $r_1 + r_2 = 2r\sin\gamma$, which is the inverse of the disc AB, in the presence of a semi-infinite line source and line sink meeting at its vertex.

It is easy to satisfy oneself that the two solutions are, in fact, the same.

The case in which the bowl becomes a disc can be derived by taking the limiting values of M and ϕ as $\gamma \to 0$, while $c \to \infty$, but it follows more directly by putting

$$M = -2\pi a\cot\omega,$$

$$\phi = \tfrac{1}{2}\log\frac{\cos(\omega-\gamma)-\cos\omega'}{\cos\omega'-\cos(\omega+\gamma)},$$

which we shall write in the forms

$$\phi = \cosh^{-1}\frac{2a}{r_1\sim r_2},$$

$$M = -\pi\sqrt{\{(r_1+r_2)^2 - 4a^2\}}.$$

These results may be regarded as furnishing the stream function and velocity potential for the disc in the presence of a semi-infinite line source and a semi-infinite line sink, in the axis, meeting at the centre of the disc, in which case ϕ must be regarded as changing sign when it passes through zero, as it does in crossing the plane of symmetry anywhere beyond the disc AB.* Or, again, they may be regarded as giving the force function and potential function in the case of the infinite plane with a circular hole under the influence of an infinite line charge in the axis. In this case ϕ is always positive.

The coordinates ω and ω' are not adapted to the solution of problems concerning the complete sphere by putting $\gamma=\pi$, for then, except in the immediate neighbourhood of the point C', they are both zero. But the value of M, corresponding to the case of the complete sphere in the presence of a diametral line source and sink, is, as is well known,

$$M=2\pi r,$$

r being the distance from the centre of the sphere. The surfaces of flow are concentric spheres, and the corresponding value of the velocity potential is

$$\phi=\log[(x+r)/y].$$

5. It is perhaps worthy of remark that if, in the equation determining ϕ as a function of ω and ω', we put

$$\phi=(\cos\omega'-\cos\gamma\cos\omega)\,F(\omega),$$

we find

$$\frac{d^2F}{d\omega^2}+2\cot\omega\frac{dF}{d\omega}-(1+2\operatorname{cosec}^2\omega)\,F=0;$$

whence $$F=[A\cot\omega+B(1-\omega\cot\omega)]\operatorname{cosec}\omega,$$

and $$\phi=[A+B(\tan\omega-\omega)]\,x/c.$$

This leads to the new potential function

$$\phi=(1-\omega\cot\omega)\,(\cos\omega'-\cos\gamma\cos\omega)/\sin\omega,$$

corresponding to which

$$M=\pi c\frac{\omega-\cos\omega\sin\omega}{\sin^4\omega}(\cos^2\omega-2\cos\gamma\cos\omega\cos\omega'+\cos^2\omega'-\sin^2\gamma).$$

* With this convention it is unnecessary to retain the ambiguity of sign before the inverse hyperbolic function in any of the values of ϕ which contain it. The change of sign is, of course, to be regarded as being from positive to negative as the point P crosses $x=0$ $(y>a)$ from right to left.

Other values of M and ϕ may be written down by remembering that, corresponding to any solution $\phi(\omega, \omega')$, there is another solution $\frac{\sin\omega}{\sin\omega'}\phi(\omega', \omega)$; and, corresponding to any value of M, $M(\omega, \omega')$, there is a another value $\frac{\sin\omega'}{\sin\omega}M(\omega', \omega)$. Further the new value of ϕ leads to a new value of M, and the new value of M leads to a new value of ϕ. There will, in general, be no limit, other than that imposed by excessive labour, to the number of values of ϕ and M which could thus be formed.

SOME DEFINITE INTEGRALS CONNECTED WITH GAUSS'S SUMS.

By *S. Ramanujan.*

1. If n is real and positive, and $|I(t)|$, where $I(t)$ is the imaginary part of t, is less than either n or 1, we have

$$(1)\quad \int_0^\infty \frac{\cos\pi tx}{\cosh\pi x}\, e^{-i\pi nx^2}\, dx$$
$$= 2\int_0^\infty\int_0^\infty \frac{\cos\pi tx\cos 2\pi xy}{\cosh\pi y}\, e^{-i\pi nx^2}\, dx\, dy$$
$$= \sqrt{n}\exp\left\{-\tfrac{1}{4}i\pi\left(1-\frac{t^2}{n}\right)\right\}\int_0^\infty \frac{\cos\pi tx}{\cosh\pi nx}\, e^{i\pi nx^2}\, dx.$$

When $n=1$ the above formula reduces to

$$(2)\quad \int_0^\infty \frac{\cos\pi tx}{\cosh\pi x}\sin\pi x^2\, dx$$
$$= \tan\{\tfrac{1}{8}\pi(1-t^2)\}\int_0^\infty \frac{\cos\pi tx}{\cosh\pi x}\cos\pi x^2\, dx.$$

If $t=0$, and

$$(3)\quad \begin{cases} \phi(n) = \displaystyle\int_0^\infty \frac{\cos\pi nx^2}{\cosh\pi x}\, dx, \\ \psi(n) = \displaystyle\int_0^\infty \frac{\sin\pi nx^2}{\cosh\pi x}\, dx, \end{cases}$$

then

$$(3')\quad \begin{cases} \phi(n) = \sqrt{\left(\frac{2}{n}\right)}\psi\left(\frac{1}{n}\right) + \psi(n), \\ \psi(n) = \sqrt{\left(\frac{2}{n}\right)}\phi\left(\frac{1}{n}\right) - \phi(n). \end{cases}$$

Similarly, if $\frac{1}{2}\sqrt{3}\,|I(t)|$ is less than either 1 or n, we have

$$(4)\qquad \int_0^\infty \frac{\cos \pi t x}{1+2\cosh(2\pi x/\sqrt{3})}\, e^{i\pi n x^2}\, dx$$

$$= \sqrt{n}\exp\left\{-\tfrac{1}{4}i\pi\left(1-\frac{t^2}{n}\right)\right\}\int_0^\infty \frac{\cos \pi t x}{1+2\cosh(2\pi n x/\sqrt{3})}\, e^{-i\pi n x^2}\, dx.$$

If in (4) we suppose $n=1$, we obtain

$$(5)\qquad \int_0^\infty \frac{\cos \pi t x \sin \pi x^2}{1+2\cosh(2\pi x/\sqrt{3})}\, dx$$

$$= \tan\{\tfrac{1}{8}\pi(1-t^2)\}\int_0^\infty \frac{\cos \pi t x \cos \pi x^2}{1+2\cosh(2\pi x/\sqrt{3})}\, dx;$$

and if $t=0$, and

$$(6)\qquad \begin{cases} \phi(n) = \displaystyle\int_0^\infty \frac{\cos \pi n x^2}{1+2\cosh(2\pi x/\sqrt{3})}\, dx, \\ \psi(n) = \displaystyle\int_0^\infty \frac{\sin \pi n x^2}{1+2\cosh(2\pi x/\sqrt{3})}\, dx, \end{cases}$$

then

$$(6')\qquad \begin{cases} \phi(n) = \sqrt{\left(\frac{2}{n}\right)}\ \psi\left(\frac{1}{n}\right) + \psi(n), \\ \psi(n) = \sqrt{\left(\frac{2}{n}\right)}\ \phi\left(\frac{1}{n}\right) - \phi(n). \end{cases}$$

In a similar manner we can prove that

$$(7)\qquad \int_0^\infty \frac{\sin \pi t x}{\tanh \pi x}\, e^{-i\pi n x^2}\, dx$$

$$= -\sqrt{n}\exp\left\{\tfrac{1}{4}i\pi\left(1+\frac{t^2}{n}\right)\right\}\int_0^\infty \frac{\sin \pi t x}{\tanh \pi n x}\, e^{i\pi n x^2}\, dx.$$

If we put $n=1$ in (7), we obtain

$$(8)\qquad \int_0^\infty \frac{\sin \pi t x}{\tanh \pi x}\cos \pi x^2\, dx$$

$$= \tan\{\tfrac{1}{8}\pi(1+t^2)\}\int_0^\infty \frac{\sin \pi t x}{\tanh \pi x}\sin \pi x^2\, dx.$$

Now

$$(9)\qquad \lim_{t\to 0}\frac{1}{t}\int_0^\infty \frac{\sin atx}{\tanh bx}\, e^{icx^2}\, dx$$

$$= \lim_{t\to 0}\frac{1}{t}\int_0^\infty \frac{2\sin atx}{e^{2bx}-1}\, e^{icx^2}\, dx + \lim_{t\to 0}\int_0^\infty \frac{\sin atx}{t}\, e^{icx^2}\, dx$$

$$= \int_0^\infty \frac{a e^{icx}}{e^{2b\sqrt{x}}-1}\, dx + \frac{ia}{2c}.$$

Hence, dividing both sides of (7) by t, and making $t \to 0$, we obtain the result corresponding to (3) and (6), viz.: if

$$(10) \qquad \begin{cases} \phi(n) = \int_0^\infty \dfrac{\cos \pi n x}{e^{2\pi\sqrt{x}} - 1}\, dx, \\ \psi(n) = \dfrac{1}{2\pi n} + \int_0^\infty \dfrac{\sin \pi n x}{e^{2\pi\sqrt{x}} - 1}\, dx, \end{cases}$$

then

$$(10') \qquad \begin{cases} \phi(n) = \dfrac{1}{n}\sqrt{\left(\dfrac{2}{n}\right)}\,\psi\left(\dfrac{1}{n}\right) - \psi(n), \\ \psi(n) = \dfrac{1}{n}\sqrt{\left(\dfrac{2}{n}\right)}\,\phi\left(\dfrac{1}{n}\right) + \phi(n). \end{cases}$$

2. I shall now show that the integral (1) may be expressed in finite terms for all rational values of n. Consider the integral

$$J(t) = \int_0^\infty \frac{\cos tx}{\cosh \frac{1}{2}\pi x}\, \frac{dx}{a^2 + x^2}.$$

If $R(a)$ and t are positive, we have

$$(11) \qquad J(t) = \frac{4}{\pi}\int_0^\infty \sum_{r=0}^{r=\infty} \frac{(-1)^r (2r+1)}{x^2 + (2r+1)^2}\, \frac{\cos tx}{a^2 + x^2}\, dx$$

$$= 2 \sum_{r=0}^{r=\infty} \frac{(-1)^r}{a^2 - (2r+1)^2} \left\{ e^{-(2r+1)t} - \frac{1}{a}(2r+1)\, e^{-at} \right\}$$

$$= \frac{\pi e^{-at}}{2a \cos \frac{1}{2}\pi a} + 2 \sum_{r=0}^{r=\infty} \frac{(-1)^r e^{-(2r+1)t}}{a^2 - (2r+1)^2},$$

and it is easy to see that this last equation remains true when t is complex, provided $R(t) > 0$ and $|I(t)| \leq \frac{1}{2}\pi$. Thus the integral $J(t)$ can be expressed in finite terms for all rational values of a. Thus, for example, we have

$$(12) \qquad \begin{cases} \int_0^\infty \dfrac{\cos tx}{\cosh \frac{1}{2}\pi x}\, \dfrac{dx}{1 + x^2} = \cosh t \log(2 \cosh t) - t \sinh t, \\ \int_0^\infty \dfrac{\cos 2tx}{\cosh \pi x}\, \dfrac{dx}{1 + x^2} = 2 \cosh t - (e^{2t} \tan^{-1} e^{-t} + e^{-2t} \tan^{-1} e^{t}), \end{cases}$$

and so on. Now let

$$(13) \qquad F(n) = \int_0^\infty \frac{\cos 2tx}{\cosh \pi x}\, e^{-i\pi n x^2}\, dx.$$

Then, if $R(a) > 0$,

$$(14) \quad \int_0^\infty e^{-an} F(n)\, dn = \int_0^\infty \frac{\cos 2tx}{\cosh \pi x} \frac{dx}{a + i\pi x^2}.$$

Now let

$$(15) \quad f(n) = \sum_{r=0}^{r=\infty} (-1)^r \exp\{-(2r+1)\, t + \tfrac{1}{4}(2r+1)^2 i\pi n\}$$

$$+ \frac{1}{\sqrt{n}} \exp\left\{-i\left(\tfrac{1}{4}\pi - \frac{t^2}{\pi n}\right)\right\} \sum_{r=0}^{r=\infty} (-1)^r \exp\left\{-(2r+1)\frac{t}{n} - \tfrac{1}{4}(2r+1)^2 \frac{i\pi}{n}\right\}.$$

Then

$$(16) \quad \int_0^\infty e^{-an} f(n)\, dn = \sum_{r=0}^{r=\infty} \frac{(-1)^r e^{-(2r+1)t}}{a - \frac{1}{4}(2r+1)^2 i\pi}$$

$$+ \sqrt{\left(\frac{\pi}{2a}\right)} \frac{\exp\{-\sqrt{(2a/\pi)}\,(1-i)\, t\}}{(1+i)\cosh\{(1+i)\sqrt{(\frac{1}{2}\pi a)}\}}$$

$$= \int_0^\infty \frac{\cos 2tx}{\cosh \pi x} \frac{dx}{a + i\pi x^2},$$

in virtue of (11); and therefore

$$(17) \quad \int_0^\infty e^{-an} \{F(n) - f(n)\}\, dn = 0.$$

Now it is known that, if $\phi(n)$ is continuous and

$$\int_0^\infty e^{-an} \phi(n)\, dn = 0,$$

for all positive values of a (or even only for an infinity of such values in arithmetical progression), then

$$\phi(n) = 0,$$

for all positive values of n. Hence

$$(18) \quad F(n) = f(n).$$

Equating the real and imaginary parts in (13) and (15) we have

$$(19) \quad \int_0^\infty \frac{\cos 2tx}{\cosh \pi x} \cos \pi n x^2\, dx$$

$$= \left\{e^{-t} \cos \frac{\pi n}{4} - e^{-3t} \cos \frac{9\pi n}{4} + e^{-5t} \cos \frac{25\pi n}{4} - \dots\right\}$$

$$+ \frac{1}{\sqrt{n}}\left\{e^{-t/n} \cos\left(\frac{\pi}{4} - \frac{t^2}{\pi n} + \frac{\pi}{4n}\right) - e^{-3t/n} \cos\left(\frac{\pi}{4} - \frac{t^2}{\pi n} + \frac{9\pi}{4n}\right) + \dots\right\},$$

$$(20)\qquad \int_0^\infty \frac{\cosh \pi x}{\cos 2tx} \sin \pi n x^2 \, dx$$

$$= -\left\{e^{-t} \sin \frac{\pi n}{4} - e^{-3t} \sin \frac{9\pi n}{4} + e^{-5t} \sin \frac{25\pi n}{4} - \ldots\right\}$$

$$+ \frac{1}{\sqrt{n}}\left\{e^{-t/n} \sin\left(\frac{\pi}{4} - \frac{t^2}{\pi n} + \frac{\pi}{4n}\right) - e^{-3t/n} \sin\left(\frac{\pi}{4} - \frac{t^2}{\pi n} + \frac{9\pi}{4n}\right) + \ldots\right\}.$$

We can verify the results (18), (19), and (20) by means of the equation (1). This equation can be expressed as a functional equation in $F(n)$, and it is easy to see that $f(n)$ satisfies the same equation.

The right-hand side of these equations can be expressed in finite terms if n is any rational number. For let $n = a/b$, where a and b are any two positive integers and one of them is odd. Then the results (19) and (20) reduce to

$$(21)\qquad 2\cosh bt \int_0^\infty \frac{\cos 2tx}{\cosh \pi x} \cos\left(\frac{\pi a x^2}{b}\right) dx$$

$$= [\cosh\{(1-b)\,t\} \cos(\pi a/4b) - \cosh\{(3-b)\,t\} \cos(9\pi a/4b)$$

$$+ \cosh\{(5-b)\,t\} \cos(25\pi a/4b) - \ldots \text{ to } b \text{ terms}]$$

$$+ \sqrt{\left(\frac{b}{a}\right)}\left[\cosh\left\{\left(1-\frac{1}{a}\right)bt\right\} \cos\left(\frac{\pi}{4} - \frac{bt^2}{\pi a} + \frac{\pi b}{4a}\right)\right.$$

$$\left. - \cosh\left\{\left(1-\frac{3}{a}\right)bt\right\} \cos\left(\frac{\pi}{4} - \frac{bt^2}{\pi a} + \frac{9\pi b}{4a}\right) + \ldots \text{ to } a \text{ terms}\right],$$

$$(22)\qquad 2\cosh bt \int_0^\infty \frac{\cos 2tx}{\cosh \pi x} \sin\left(\frac{\pi a x^2}{b}\right) dx$$

$$= -[\cosh\{(1-b)\,t\} \sin(\pi a/4b) - \cosh\{(3-b)\,t\} \sin(9\pi a/4b)$$

$$+ \cosh\{(5-b)\,t\} \sin(25\pi a/4b) - \ldots \text{ to } b \text{ terms}]$$

$$+ \sqrt{\left(\frac{b}{a}\right)}\left[\cosh\left\{\left(1-\frac{1}{a}\right)bt\right\} \sin\left(\frac{\pi}{4} - \frac{bt^2}{\pi a} + \frac{\pi b}{4a}\right)\right.$$

$$\left. - \cosh\left\{\left(1-\frac{3}{a}\right)bt\right\} \sin\left(\frac{\pi}{4} - \frac{bt^2}{\pi a} + \frac{9\pi b}{4a}\right) + \ldots \text{ to } a \text{ terms}\right].$$

Thus, for example, we have, when $a = 1$ and $b = 1$,

$$(23)\qquad \int_0^\infty \frac{\cos \pi x^2}{\cosh \pi x} \cos 2\pi tx \, dx = \frac{1 + \sqrt{2} \sin \pi t^2}{2\sqrt{2} \cosh \pi t},$$

$$(24)\qquad \int_0^\infty \frac{\sin \pi x^2}{\cosh \pi x} \cos 2\pi tx \, dx = \frac{-1 + \sqrt{2} \cos \pi t^2}{2\sqrt{2} \cosh \pi t}.$$

It is easy to verify that (23) and (24) satisfy the relation (2).

The values of the integrals

$$\int_0^\infty \frac{\cos \pi n x^2}{\cosh \pi x}\, dx, \qquad \int_0^\infty \frac{\sin \pi n x^2}{\cosh \pi x}\, dx$$

can be obtained easily from the preceding results by putting $t = 0$, and need no special discussion. By successive differentiations of the results (19) and (20) with respect to t and n, we can evaluate the integrals

$$(25) \qquad \begin{cases} \displaystyle\int_0^\infty x^{2m-1} \frac{\sin tx}{\cosh \pi x} \begin{matrix}\cos \\ \sin\end{matrix} \pi n x^2\, dx, \\ \displaystyle\int_0^\infty x^{2m} \frac{\cos tx}{\cosh \pi x} \begin{matrix}\cos \\ \sin\end{matrix} \pi n x^2\, dx, \end{cases}$$

for all rational values of n and all positive integral values of m. Thus, for example, we have

$$(26) \qquad \begin{cases} \displaystyle\int_0^\infty x^2 \frac{\cos \pi x^2}{\cosh \pi x}\, dx = \frac{1}{8\sqrt{2}} - \frac{1}{4\pi}, \\ \displaystyle\int_0^\infty x^2 \frac{\sin \pi x^2}{\cosh \pi x}\, dx = \frac{1}{8} - \frac{1}{8\sqrt{2}}. \end{cases}$$

3. We can get many interesting results by applying the theory of Cauchy's reciprocal functions to the preceding results. It is known that, if

$$(27) \qquad \int_0^\infty \phi(x) \cos knx\, dx = \psi(n),$$

then

(27) (i) $\quad \frac{1}{2}\alpha \{\frac{1}{2}\phi(0) + \phi(\alpha) + \phi(2\alpha) + \phi(3\alpha) + \ldots\}$
$$= \tfrac{1}{2}\psi(0) + \psi(\beta) + \psi(2\beta) + \psi(3\beta) \ldots,$$
with the condition $\alpha\beta = 2\pi/k$;

(27) (ii) $\quad \alpha\sqrt{2}\{\phi(\alpha) - \phi(3\alpha) - \phi(5\alpha) + \phi(7\alpha) + \phi(9\beta) - \ldots\}$
$$= \psi(\beta) - \psi(3\beta) - \psi(5\beta) + \psi(7\beta) + \psi(9\beta) - \ldots,$$
with the condition $\alpha\beta = \pi/4k$;

(17) (iii) $\quad \alpha\sqrt{3}\{\phi(\alpha) - \phi(5\alpha) - \phi(7\alpha) + \phi(11\alpha) + \phi(13\alpha) - \ldots\}$
$$= \psi(\beta) - \psi(5\beta) - \psi(7\beta) + \psi(11\beta) + \psi(13\beta) - \ldots,$$
with the condition $\alpha\beta = \pi/6k$, where 1, 5, 7, 11, 13, ... are the odd natural numbers without the multiples of 3.

There are of course corresponding results for the function

$$(28) \qquad \int_0^\infty \phi(x) \sin knx\, dx = \psi(n),$$

such as

$$\alpha\{\phi(\alpha) - \phi(3\alpha) + \phi(5\alpha) - \ldots\} = \psi(\beta) - \psi(3\beta) + \psi(5\beta) - \ldots$$

with the condition $\alpha\beta = \pi/2k$.

Thus from (23) and (27) (i) we obtain the following result. If

$$(29) \qquad F(\alpha, \beta) = \sqrt{\alpha}\left\{\tfrac{1}{2} + \sum_{r=1}^{r=\infty} \frac{\cos r^2\pi\alpha^2}{\cosh r\pi\alpha}\right\} - \sqrt{\beta}\sum_{r=1}^{r=\infty} \frac{\sin r^2\pi\beta^2}{\cosh r\pi\beta},$$

then

$$F(\alpha, \beta) = F(\beta, \alpha) = \sqrt{(2\alpha)}\left\{\tfrac{1}{2} + e^{-\pi\alpha} + e^{-4\pi\alpha} + e^{-9\pi\alpha} + \ldots\right\},$$

provided that $\alpha\beta = 1$,

4. If, instead of starting with the integral (11), we start with the corresponding sine integral, we can show that, when $R(a)$ and $R(t)$ are positive and $|I(t)| \leq \pi$,

$$(30) \qquad \int_0^\infty \frac{\sin tx}{\sinh \pi x} \frac{dx}{a^2 + x^2} = \frac{1}{2a^2} - \frac{\pi e^{-at}}{2a \sin \pi a} + \sum_{r=1}^{r=\infty} \frac{(-1)^r e^{-rt}}{a^2 - r^2}.$$

Hence the above integral can be expressed in finite terms for all rational values of a. For example, we have

$$(31) \qquad \int_0^\infty \frac{\sin tx}{\sinh \tfrac{1}{2}\pi x} \frac{dx}{1 + x^2} = e^t \tan^{-1} e^{-t} - e^{-t} \tan^{-1} e^t.$$

From (30) we can deduce that

$$(32) \qquad \int_0^\infty \frac{\sin 2tx}{\sinh \pi x} e^{-i\pi n x^2}\, dx$$

$$= \tfrac{1}{2} - e^{-2t + i\pi n} + e^{-4t + 4i\pi n} - e^{-6t + 9i\pi n} + \ldots$$

$$- \frac{1}{\sqrt{n}} \exp\left\{\left(\tfrac{1}{4}\pi + \frac{t^2}{\pi n}\right) i\right\}\left\{e^{-(t + \frac{1}{4}i\pi)^2/n} + e^{-(3t + \frac{9}{4}i\pi)^2/n} + \ldots\right\},$$

$R(t)$ being positive and $|I(t)| \leq \tfrac{1}{2}\pi$. The right-hand side can be expressed in finite terms for all rational values of n.

Thus, for example, we have

$$(33)\qquad \int_0^\infty \frac{\cos \pi x^2}{\sinh \pi x} \sin 2\pi t x \, dx = \frac{\cosh \pi t - \cos \pi t^2}{2 \sinh \pi t},$$

$$(34)\qquad \int_0^\infty \frac{\sin \pi x^2}{\sinh \pi x} \sin 2\pi t x \, dx = \frac{\sin \pi t^2}{2 \sinh \pi t},$$

and so on.

Applying the formula (28) to (33) and (34), we have, when $\alpha\beta = \frac{1}{4}$,

$$(35)\qquad \begin{cases} \sqrt{\alpha} \sum_{r=0}^{r=\infty} (-1)^r \dfrac{\cos\{(2r+1)^2 \pi\alpha^2\}}{\sinh\{(2r+1)\pi\alpha\}} \\ \qquad + \sqrt{\beta} \sum_{r=0}^{r=\infty} (-1)^r \dfrac{\cos\{(2r+1)^2 \pi\beta^2\}}{\sinh\{(2r+1)\pi\beta\}} \\ \qquad = 2\sqrt{\alpha}\{\tfrac{1}{2} + e^{-2\pi\alpha} + e^{-8\pi\alpha} + e^{-18\pi\alpha} + \ldots\}^2; \\ \sqrt{\alpha} \sum_{r=0}^{r=\infty} (-1)^r \dfrac{\sin\{(2r+1)^2 \pi\alpha^2\}}{\sinh\{(2r+1)\pi\alpha\}} \\ \qquad = \sqrt{\beta} \sum_{r=0}^{r=\infty} (-1)^r \dfrac{\sin\{(2r+1)^2 \pi\beta^2\}}{\sinh\{(2r+1)\pi\beta\}}. \end{cases}$$

By successive differentiation of (32) with respect to t and n, we can evaluate the integrals

$$(36)\qquad \begin{cases} \int_0^\infty x^{2m-1} \dfrac{\sin tx}{\sinh \pi x} \dfrac{\cos}{\sin} \pi n x^2 \, dx, \\ \int_0^\infty x^{2m} \dfrac{\sin tx}{\sinh \pi x} \dfrac{\cos}{\sin} \pi n x^2 \, dx \end{cases}$$

for all rational values of n and all positive integral values of m. Thus, for example, we have

$$(37)\qquad \begin{cases} \int_0^\infty x \dfrac{\cos \pi x^2}{\sinh \pi x} dx = \dfrac{1}{8}, \quad \int_0^\infty x \dfrac{\sin \pi x^2}{\sinh \pi x} dx = \dfrac{1}{4\pi}, \\ \int_0^\infty x^3 \dfrac{\cos \pi x^2}{\sinh \pi x} dx = \dfrac{1}{16}\left(\dfrac{1}{4} - \dfrac{3}{\pi^2}\right), \quad \int_0^\infty x^3 \dfrac{\sin \pi x^2}{\sinh \pi x} dx = \dfrac{1}{16\pi}, \end{cases}$$

and so on.

The denominators of the integrands in (25) and (36) are $\cosh \pi x$ and $\sinh \pi x$. Similar integrals having the denominator of their integrands equal to

$$\prod_1^r \cosh(\pi a_r x) \sinh(\pi b_r x)$$

can be evaluated, if a_r and b_r are rational, by splitting up the integrand into partial fractions.

5. The preceding formulæ may be generalized. Thus it may be shewn that, if $R(a)$ and $R(t)$ are positive, $|I(t)| \leq \pi$, and $-1 < R(\theta) < 1$, then

$$(38) \qquad \sin \pi\theta \int_0^\infty \frac{\cos tx}{\cosh \pi x + \cos \pi\theta} \frac{dx}{a^2 + x^2}$$

$$= \frac{\pi}{2a} \frac{e^{-at} \sin \pi\theta}{\cos \pi a + \cos \pi\theta}$$

$$+ \sum_{r=0}^{r=\infty} \left\{ \frac{e^{-(2r+1-\theta)t}}{a^2 - (2r+1-\theta)^2} - \frac{e^{-(2r+1+\theta)t}}{a^2 - (2r+1+\theta)^2} \right\}.$$

From (38) it can be deduced that, if n and $R(t)$ are positive, $|I(t)| \leq \pi$, and $-1 < \theta < 1$, then

$$(39) \qquad \sin \pi\theta \int_0^\infty \frac{\cos tx}{\cosh \pi x + \cos \pi\theta} e^{-i\pi n x^2} dx$$

$$= \sum_{r=0}^{r=\infty} \{ e^{-(2r+1-\theta)t + (2r+1-\theta)^2 i\pi n} - e^{-(2r+1+\theta)t + (2r+1+\theta)^2 i\pi n} \}$$

$$+ \frac{1}{\sqrt{n}} \exp\left\{ -\frac{1}{4} i \left(\pi - \frac{t^2}{\pi n} \right) \right\} \sum_{r=1}^{r=\infty} (-1)^{r-1} \sin r\pi\theta \, e^{-(2rt + r^2 i\pi)/4n}.$$

The right-hand side can be expressed in finite terms if n and θ are rational. In particular, when $\theta = \frac{1}{2}$, we have

$$(40) \qquad \int_0^\infty \frac{\cos tx}{1 + 2\cosh(2\pi x/\sqrt{3})} e^{-i\pi n x^2} dx$$

$$= \tfrac{1}{2} \{ e^{-(t\sqrt{3} - i\pi n)/3} - e^{-(2t\sqrt{3} - 4i\pi n)/3} + e^{-(4t\sqrt{3} - 16i\pi n)/3} - \ldots \}$$

$$+ \frac{1}{2\sqrt{n}} \exp\left\{ -\frac{1}{4} i \left(\pi - \frac{t^2}{\pi n} \right) \right\} \{ e^{-(t\sqrt{3} + i\pi)/3n} - e^{-(2t\sqrt{3} + 4i\pi)/3n}$$

$$+ e^{-(4t\sqrt{3} + 16i\pi)/3n} - \ldots \},$$

where 1, 2, 4, 5, ... are the natural numbers without the multiples of 3.

As an example, when $n = 1$, we have

$$(41) \quad \begin{cases} \displaystyle\int_0^\infty \frac{\cos \pi x^2 \cos \pi tx}{1 + 2\cosh(2\pi x/\sqrt{3})} dx = \frac{1 - 2\sin\{(\pi - 3\pi t^2)/12\}}{8\cosh(\pi t/\sqrt{3}) - 4}, \\ \displaystyle\int_0^\infty \frac{\sin \pi x^2 \cos \pi tx}{1 + 2\cosh(2\pi x/\sqrt{3})} dx = \frac{-\sqrt{3} + 2\cos\{(\pi - 3\pi t^2)/12\}}{8\cosh(\pi t/\sqrt{3}) - 4}. \end{cases}$$

6. The formula (32) assumes a neat and elegant form when t is changed to $t+\frac{1}{2}i\pi$. We have then

$$(42)\qquad \int_0^\infty \frac{\sin tx}{\tanh \pi x}\, e^{-i\pi n x^2}\, dx \quad (n>0,\ t>0)$$

$$= \{\tfrac{1}{2} + e^{-t+i\pi n} + e^{-2t+4i\pi n} + e^{-3t+9i\pi n} + \ldots\}$$

$$- \frac{1}{\sqrt{n}} \exp\left\{-\frac{1}{4}\, i\left(\pi + \frac{t^2}{\pi n}\right)\right\} \{\tfrac{1}{2} + e^{-(t+i\pi)^2/n} + e^{-(2t+4i\pi)^2/n} + \ldots\}.$$

In particular, when $n=1$, we have

$$(43)\qquad \begin{cases} \displaystyle\int_0^\infty \frac{\cos \pi x^2}{\tanh \pi x} \sin 2\pi t x\, dx = \tfrac{1}{2} \tanh \pi t \{1 - \cos(\tfrac{1}{4}\pi + \pi t^2)\}, \\ \displaystyle\int_0^\infty \frac{\cos \pi x^2}{\tanh \pi x} \sin 2\pi t x\, dx = \tfrac{1}{2} \tanh \pi t \sin(\tfrac{1}{4}\pi + \pi t^2). \end{cases}$$

We shall now consider an important special case of (42). It can easily be seen from (9) that the left-hand side of (42), when divided by t, tends to

$$(44)\qquad \int_0^\infty \frac{\cos \pi n x}{e^{2\pi\sqrt{x}} - 1}\, dx - i\left\{\frac{1}{2\pi n} + \int_0^\infty \frac{\sin \pi n x}{e^{2\pi\sqrt{x}} - 1}\, dx\right\}$$

as $t \to 0$. But the limit of the right-hand side of (42) divided by t can be found when n is rational. Let then $n = a/b$, where a and b are any two positive integers, and let

$$\phi(n) = \int_0^\infty \frac{\cos \pi n x}{e^{2\pi\sqrt{x}} - 1}\, dx, \quad \psi(n) = \frac{1}{2\pi n} + \int_0^\infty \frac{\sin \pi n x}{e^{2\pi\sqrt{x}} - 1}\, dx.$$

The relation between $\phi(n)$ and $\psi(n)$ has been stated already in (10′). From (42) and (44) it can easily be deduced that, if a and b are both odd, then

$$(45)\qquad \begin{cases} \displaystyle\phi\left(\frac{a}{b}\right) = \tfrac{1}{4} \sum_{r=1}^{r=b} (b-2r) \cos\left(\frac{r^2\pi a}{b}\right) \\ \qquad\qquad \displaystyle - \frac{b}{4a} \sqrt{\left(\frac{b}{a}\right)} \sum_{r=1}^{r=a} (a-2r) \sin\left(\tfrac{1}{4}\pi + \frac{r^2 b\pi}{a}\right), \\ \displaystyle\psi\left(\frac{a}{b}\right) = -\tfrac{1}{4} \sum_{r=1}^{r=b} (b-2r) \sin\left(\frac{r^2\pi a}{b}\right) \\ \qquad\qquad \displaystyle + \frac{b}{4a} \sqrt{\left(\frac{b}{a}\right)} \sum_{r=1}^{r=a} (a-2r) \cos\left(\tfrac{1}{4}\pi + \frac{r^2\pi b}{a}\right). \end{cases}$$

It can easily be seen that these satisfy the relation (10′). Similarly, when one of a and b is odd and the other even, it can be shown that

$$(46)\quad \begin{cases} \phi\left(\frac{a}{b}\right) = \frac{\sigma}{4\pi a\sqrt{a}} - \frac{1}{2}\sum_{r=1}^{r=b} r\left(1-\frac{r}{b}\right)\cos\left(\frac{r^2\pi a}{b}\right) \\ \qquad + \frac{b}{2a}\sqrt{\left(\frac{b}{a}\right)}\sum_{r=1}^{r=a} r\left(1-\frac{r}{a}\right)\sin\left(\frac{1}{4}\pi + \frac{r^2\pi b}{a}\right), \\ \psi\left(\frac{a}{b}\right) = \frac{\sigma'}{4\pi a\sqrt{a}} + \frac{1}{2}\sum_{r=1}^{r=b} r\left(1-\frac{r}{b}\right)\sin\left(\frac{r^2\pi a}{b}\right) \\ \qquad - \frac{b}{2a}\sqrt{\left(\frac{b}{a}\right)}\sum_{r=1}^{r=a} r\left(1-\frac{r}{a}\right)\cos\left(\frac{1}{4}\pi + \frac{r^2\pi b}{a}\right). \end{cases}$$

where

$$(47)\quad \begin{cases} \sigma = \sqrt{b}\sum_{1}^{a}\cos\left(\frac{1}{4}\pi + \frac{r^2\pi b}{a}\right) = \sqrt{a}\sum_{1}^{b}\sin\left(\frac{r^2\pi a}{b}\right), \\ \sigma' = \sqrt{b}\sum_{1}^{a}\sin\left(\frac{1}{4}\pi + \frac{r^2\pi b}{a}\right) = \sqrt{a}\sum_{1}^{b}\cos\left(\frac{r^2\pi a}{b}\right). \end{cases}$$

Thus, for example, we have

$$(48)\quad \begin{cases} \phi(0) = \frac{1}{16}, \quad \phi(1) = \frac{2-\sqrt{2}}{8}, \quad \phi(2) = \frac{1}{16}, \quad \phi(4) = \frac{3-\sqrt{2}}{32}, \\ \phi(6) = \frac{13-4\sqrt{3}}{144}, \quad \phi\left(\frac{1}{2}\right) = \frac{1}{4\pi}, \quad \phi\left(\frac{2}{5}\right) = \frac{8-3\sqrt{5}}{16}, \end{cases}$$

and so on.

By differentiating (42) with respect to n, we can evaluate the integrals

$$(49)\quad \int_0^\infty \frac{x^m}{e^{2\pi\sqrt{x}}-1}\begin{matrix}\cos\\ \sin\end{matrix}\pi n x\,dx$$

for all rational values of n and positive integral values of m. Thus, for example, we have

$$(50)\quad \begin{cases} \int_0^\infty \frac{x\cos\frac{1}{2}\pi x}{e^{2\pi\sqrt{x}}-1}\,dx = \frac{13-4\pi}{8\pi^2}, \\ \int_0^\infty \frac{x\cos 2\pi x}{e^{2\pi\sqrt{x}}-1}\,dx = \frac{1}{64}\left(\frac{1}{2}-\frac{3}{\pi}+\frac{5}{\pi^2}\right), \\ \int_0^\infty \frac{x^2\cos 2\pi x}{e^{2\pi\sqrt{x}}-1}\,dx = \frac{1}{256}\left(1-\frac{5}{\pi}+\frac{5}{\pi^2}\right), \end{cases}$$

and so on.

Note by Mr. G. H. Hardy on the preceding paper.

The integral (1) was evaluated, in the special case when $t=0$ and n is the reciprocal of an odd integer, by Kronecker (*Crelle's Journal*, vol. 55, p. 345). An equation substantially equivalent to the equations (3′) was given by me (*Quarterly Journal*, vol. 35, p. 204) and applied to the evaluation of (1) when $t=0$ and n is an integer or the reciprocal of an integer, odd or even. The remaining formulæ of the paper, so far as I know, are new.

The theorem appealed to in the inference of (18) from (17) seems to be due to Lerch (*Acta Mathematica*, vol. 27, p. 347). The equivalence of the two forms of σ and σ' given in (47) was first proved by Schaar: see Lindelöf, *Le calcul des résidus*, p. 75.

ON CERTAIN TRIANGLES AND PARALLELOGRAMS WHICH HAVE THE SAME CENTROID.

By *H. Holden.*

Take any triangle ABC, produce each of the sides, taken in order, to a distance equal to p times its own length, and join up the ends of these lines. We thus get the triangle $X_1Y_1Z_1$ or $X_2Y_2Z_2$, according to the direction of travelling round ABC. If then we travel round $X_1Y_1Z_1$ or $X_2Y_2Z_2$ in the opposite direction to that by which it was derived from ABC, and produce each side by p times its own length, we get the triangle PQR in each case.

Figure 1 shows the simplest case, where $p=1$. The following properties may then be easily proved either by similar triangles or by the coordinate geometry method used later in discussing the more general case of rotation as well as extension. The results given below refer mainly to the special case figured, but some of the general values or extension have also been worked out.

(1) All the triangles have the same centroid, which lies on the straight lines joining corresponding corners of the triangles PQR and ABC. These straight lines pass through the points of intersection of the XYZ triangles.

(2) The sides of the triangle PQR are parallel to those of ABC, and generally $3p^2+3p+1$ times as long.

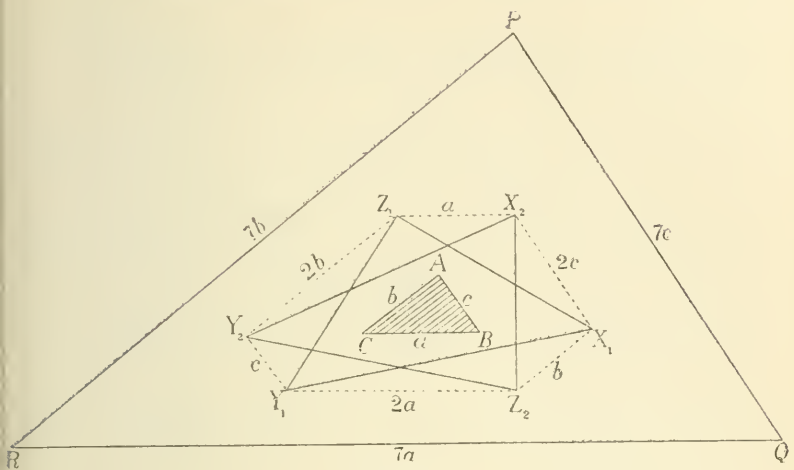

Figure 1.

(3) Each side of the triangles $X_1Y_1Z_1$ and $X_2Y_2Z_2$ is divided into three parts, whose lengths are as

$$3p^2 + 4p + 1 : 3p^2 + 3p + 1 : 3p^2 + 2p,$$

if we travel round each triangle in the same direction as that by which it was derived from ABC. This fact points to the following proposition: If the sides of any triangle $X_1Y_1Z_1$, taken in order, be divided in the ratio of

$$3p^2 + 4p + 1 : 3p^2 + 3p + 1 : 3p^2 + 2p,$$

and the last division on each side be joined to the first division of the next so as to form the triangle $X_2Y_2Z_2$, the sides of this triangle, taken in the opposite order, will be divided in the same ratio.

(4) The lines RX_2Y_2 and RY_1X_1 will, if produced, trisect PQ, and similarly for the other sides. More generally, PQ is thus divided into parts having the ratio $p:1:p$. This gives a simple method of reproducing the interior triangles from PQR.

(5) The sides of the triangle ABC, if produced beyond X_1, X_2, etc., by their own length, will end on the sides of PQR.

(6) The sides of the dotted hexagon, if produced their own length, will end on the sides of PQR.

All quadrilaterals do not develop into similar quadrilaterals, but a parallelogram does. The sides of the final parallelogram

are parallel to those of the original one, and $2p^2+2p+1$ times as long. The sides of the two intermediate parallelograms are divided in the ratio of $2p^3+2p : 2p^2+2p+1 : 2p+1$.

Extension and rotation.

Let the coordinates of A, B, C, referred to rectangular axes, be x_1, y_1; x_2, y_2; and x_3, y_3. Produce AB by p times its own length; then turn anti-clockwise relative to AB through a right-angle, and proceed along this new direction to a distance of q times AB. Then if Z be the point reached, its coordinates are

$$Z_x=x_3+p(x_2-x_1)-q(y_2-y_1),\quad Z_y=y_2+p(y_2-y_1)+q(x_2-x_1),$$

operating similarly on BC and CA, we get to the points X and Y, given by

$$X_x=x_3+p(x_3-x_2)-q(y_3-y_2),\quad X_y=y_3+p(y_3-y_2)+q(x_3-x_2),$$

$$Y_x=x_1+p(x_1-x_3)-q(y_1-y_3),\quad Y_y=y_1+p(y_1-y_3)+q(x_1-x_3).$$

As $X_x+Y_x+Z_x=x_1+x_2+x_3$, and similarly for the y coordinates, ABC, and XYZ, and all subsequent triangles formed by the same operation, have the same centroid.

Travelling round XYZ in the opposite direction to that by which it was derived, namely, from Z to Y, and so on, but again rotating in an anti-clockwise direction relative to ZY, etc., we get to points P, Q, R, whose coordinates are

$$P_x=x_1+(p+p^2-q^2)(2x_1-x_2-x_3)-q(2p+1)(2y_1-y_2-y_3),$$

$$P_y=y_1+(p+p^2-q^2)(2y_1-y_2-y_3)+q(2p+1)(2x_1-x_2-x_3).$$

As these values are unaltered by interchanging x_2 and x_3, y_2 and y_3, reversing the direction of travel round ABC and XYZ, would give the same final triangle PQR.

If $\bar{x}$, $\bar{y}$, be the coordinates of the centroid of A, B, C, we have

$$P_x=x_1+3(p+p^2-q^2)(x_1-\bar{x})-3q(2p+1)(y_1-\bar{y}),$$

$$P_y=y_2+3(p+p^2-q^2)(y_1-\bar{y})+3q(2p+1)(x_1-\bar{x}),$$

which shows that P, Q, R may be reached by a single operation on the lines joining the centroid to the corners of ABC. If $q=0$ or $p=-\frac{1}{2}$, P, A, and the centroid are collinear.

Again

$$P_x - Q_x = (3p^2 + 3p + 1 - 3q^2)(x_1 - x_2) - 3q(2p+1)(y_1 - y_2),$$

$$P_y - Q_y = (3p^2 + 3p + 1 - 3q^2)(y_1 - y_2) + 3q(2p+1)(x_1 - x_2).$$

By altering the sign of q throughout, we get the effect of a clockwise rotation.

The following special cases may be noticed:

(1) If $q = 0$ or $p = -\frac{1}{2}$,

$$\frac{P_y - Q_y}{P_x - Q_x} = \frac{y_1 - y_2}{x_1 - x_2},$$

so that the sides of PQR are parallel to those of ABC. Hence, if ABC is a straight line, so also is PQR. Taking the x axis along ABC, we get $P_y = 0$, and so PQR lies along ABC. In the same way it may be proved that any straight line $ABCD\ldots Z$ is developed by operating as before on the parts AB, CB, CD, ..., ZA, and then similarly on the resultant polygon into a straight line lying along $ABCD\ldots Z$.

If a (p, q) operation on $ABC\ldots Z$ be followed by an (r, q) operation on the resultant polygon, we find that $P_y = 0$, if $p + r + 1 = 0$; which shows that instead of the mid-points, we may take any corresponding points on the sides from which to draw the q perpendiculars. As then p and q may have any values, we have the following proposition: If we rotate, with proportional extension, each part (including ZA) of a straight line $ABCD\ldots Z$ through any constant angle about its nearer end, and then rotate about their further ends with the same proportional extension the sides of the resultant polygon through the same angles, the points reached lie on $ABC\ldots Z$.

(2) Taking again $ABC\ldots Z$, a straight line along the x axis, we have, for all values of p and q,

$$\frac{P_y - Q_y}{P_x - Q_x} = \frac{3q(2p+1)}{3p^2 + 3p + 1 - 3q^2} = \frac{Q_y - R_y}{Q_x - R_x} = \text{etc.},$$

and hence, if we rotate with proportional extension each part of a straight line $ABCD\ldots Z$ through any constant angle about its near end, and then rotate (with the same proportional extension) about their nearer ends, the sides of the resultant polygon through the same angle, the points reached lie on a straight line inclined to $ABCD\ldots Z$.

(3) If

$$p = -\tfrac{1}{2}, \quad q = \pm \frac{1}{2\sqrt{3}}, \quad p + p^2 - q^2 = -\tfrac{1}{3},$$

and $$P_x = x_1 - (x_1 - \bar{x}) = \bar{x},$$
$$P_y = \bar{y},$$

and so the triangle PQR reduces to a point: the centroid of ABC. As the above operation consists in finding the centroid of an equilateral triangle described on the line operated upon, it follows that the XYZ triangles must also be equilateral. Hence the proposition: If equilateral triangles be described, inwards or outwards, on the sides of any triangle, their centroids form an equilateral triangle (Fig. 2). This holds if, instead of the original triangle, any straight line divided into any two parts be operated on as above. The areas of the two equilateral triangles so formed have for their sum the average area of the equilateral triangles described on the sides of ABC, and for their difference the area of ABC.

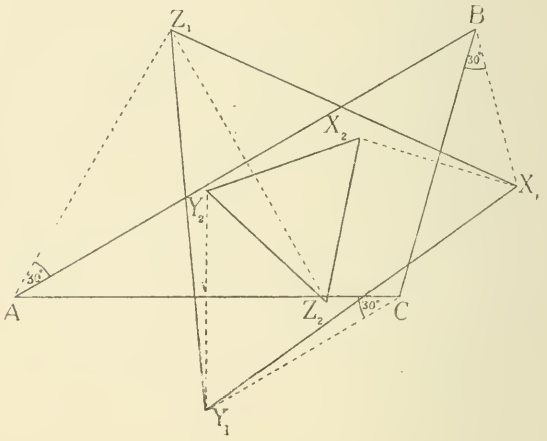

Figure 2.

More generally, if equilateral triangles be described inwards or outwards on the sides of any polygon, and on the sides of the polygon formed by their centroids equilateral triangles be described inwards, the centroids of these latter triangles coincide with those of the triangles formed by joining alternate corners of the original polygon.

(4) If $p = -\frac{1}{2}$, $q = \pm\frac{1}{2}$, $p + p^2 - q^2 = -\frac{1}{2}$,

and $$P_x = x_1 - \tfrac{1}{2}(2x_1 - x_2 - x_3) = \tfrac{1}{2}(x_2 + x_3),$$
$$P_y = \tfrac{1}{2}(y_2 + y_3).$$

Hence, if squares be described, outwards or inwards, on the sides of any triangle, and on the sides of the triangle formed by their centroids squares be described, inwards or outwards, the centroids of these latter squares coincide with the mid-points of the sides of the original triangle. The original triangle may be a straight line.

More generally, starting with any polygon and performing the above double operation, the final points reached are the mid-points of the lines joining alternate corners of the polygon.

(5) If $p = -\frac{1}{2}$, $q = \pm\frac{\sqrt{3}}{2}$, $p + p^2 - q^2 = -1$, and

$$P_x = x_1 - 2x_1 + x_2 + x_3.$$

Similarly $$Q_x = x_2 - 2x_2 + x_1 + x_3.$$

Therefore $$P_x + Q_x = 2x_3,$$

and similarly $$P_y + Q_y = 2y_3.$$

Hence, if on the sides of any triangle ABC, equilateral triangles be described, outwards or inwards, and on the sides of the triangle formed by their vertices, equilateral triangles are described, inwards or outwards, their vertices form a triangle of which A, B, C are the mid-points of the sides. The triangle ABC may be a straight line.

(6) If

$$p = -\tfrac{1}{3}, \quad q = \pm\tfrac{1}{3}, \quad 3p^2 + 3p + 1 - 3q^2 = 0,$$

and $$3q(2p+1) = \pm\tfrac{1}{3},$$

and $$P_x - Q_x = \mp\tfrac{1}{3}(y_1 - y_2),$$

$$P_y - Q_y = \pm\tfrac{1}{3}(x_1 - x_2).$$

If $$p = -\tfrac{2}{3}, \quad q = \pm\tfrac{1}{3}, \quad 3p^2 + 3p + 1 - 3q^2 = 0,$$

and $$3q(2p+1) = \mp\tfrac{1}{3},$$

and $$P_x - Q_x = \pm\tfrac{1}{3}(y_1 - y_2),$$

$$P_y - Q_y = \mp\tfrac{1}{3}(x_1 - x_2).$$

Hence, if from corresponding points of trisection of the sides of a triangle perpendiculars are drawn, inwards or outwards, whose lengths are one-third of their respective sides, and if the resulting triangle be similarly operated on, according to the rule given above, the final triangle will have sides one-third of those of the original triangle and perpendicular to them. The original triangle may be a straight line.

Parallelograms.

Let D be the point x_4, y_4. Operating on AB, BC, CD, DA to get W, X, Y, Z respectively, we have

$$W_x = x_2 + p(x_2 - x_1) - q(y_2 - y_1), \quad W_y = y_2 + p(y_2 - y_1) + q(x_2 - x_1),$$

$$X_x = x_3 + p(x_3 - x_2) - q(y_3 - y_2), \quad X_y = y_3 + p(y_3 - y_2) + q(x_3 - x_2),$$

$$Y_x = x_4 + p(x_4 - x_3) - q(y_4 - y_3), \quad Y_y = y_4 + p(y_4 - y_3) + q(x_4 - x_3),$$

$$Z_x = x_1 + p(x_1 - x_4) - q(y_1 - y_4), \quad Z_y = y_1 + p(y_1 - y_4) + q(x_1 - x_4).$$

Hence $WXYZ$, and all figures subsequently formed by similar operations, have the same centroid as $ABCD$.

If $ABCD$ is a parallelogram,

$$x_2 - x_1 = x_3 - x_4,$$

$$y_2 - y_1 = y_3 - y_4,$$

and so

$$\tfrac{1}{2}(W_x + Y_x) = \tfrac{1}{2}(x_2 + x_4); \quad \tfrac{1}{2}(W_y + Y_y) = \tfrac{1}{2}(y_2 + y_4).$$

Hence WY and similarly XZ, AC, and BD bisect one another, and so $WXYZ$ is a parallelogram.

Operating on WZ, ZY, YX, XW to get P, Q, R, S respectively, we have

$$P_x = x_1 + (p + p^2 - q^2)(2x_1 - x_2 - x_4) - q(2p + 1)(2y_1 - y_2 - y_4).$$

But for a parallelogram

$$2x_1 - x_2 - x_4 = x_1 - x_3 = 2(x_1 - \bar{x}),$$

$$2y_1 - y_2 - y_4 = y_1 - y_3 = 2(y_1 - \bar{y}).$$

Hence

$$P_x = x_1 + 2(p + p^2 - q^2)(x_1 - \bar{x}) - 2q(2p + 1)(y_1 - \bar{y}),$$

$$P_y = y_1 + 2(p + p^2 - q^2)(y_1 - \bar{y}) + 2q(2p + 1)(x_1 - \bar{x}),$$

which shows that $PQRS$ may be produced by a single operation on the lines joining the centroid of $ABCD$ to its corners. Again

$$P_x - Q_x = (x_1 - x_2)(2p^2 + 2p + 1 - 2q^2) - 2q(2p + 1)(y_1 - y_2),$$

$$P_y - Q_y = (y_1 - y_2)(2p^2 + 2p + 1 - 2q^2) + 2q(2p + 1)(x_1 - x_2).$$

Hence the sides of $PQRS$ and $ABCD$ are parallel if $q = 0$ or $p = -\frac{1}{2}$. In the former case the sides of $PQRS$ are $2p^2 + 2p + 1$ times, and in the latter case $(\frac{1}{2} - 2q^2)$ times, as long as those of $ABCD$. The sides of $PQRS$ and $ABCD$

are at right-angles if p and q satisfy $2p^2+2p+1-2q^2=0$, *e.g.*, if $p=\frac{1}{6}$, $q=\pm\frac{5}{6}$; $p=-\frac{1}{8}$, $q=\pm\frac{3}{8}$, etc.

Two special cases may be noticed:

(1) If $p=-\frac{1}{2}$, $q=\pm\frac{1}{2}$, we have $p+p^2-q^2=-\frac{1}{2}$, and so

$$P_x=\bar{x}, \quad P_y=\bar{y}.$$

Hence $PQRS$ reduces to a point, and $WXYZ$ must be a square, which gives the following proposition: If on the sides of a parallelogram squares be described, inwards or outwards, their centroids are the corners of a square. The areas of the two squares so formed have for their sum twice the average area of the squares on the sides of $ABCD$, and for their difference twice the area of $ABCD$.

(2) If $p=-\frac{1}{2}$, $q=\pm\frac{\sqrt{3}}{2}$, $p+p^2-q^2=-1$, and so

$$P_x=x_1-2(x_1-\bar{x})=x_1-(x_1-x_3)=x_3,$$
$$P_y=y_3.$$

Hence the proposition: If on the sides of a parallelogram equilateral triangles be described, outwards or inwards, and on the sides of the parallelogram formed by their vertices, equilateral triangles be described, inwards or outwards, the parallelogram formed by the new vertices coincides with the original parallelogram.

These propositions hold for a straight line $ABDC$ if $AB=DC$, and the parts are taken as AB, BC, CD, and DA.

Rotation about the mid-points of the sides.

Drawing from the mid-point a line n times the side and inclined to it at an angle θ is equivalent to putting

$$p=n\cos\theta-\tfrac{1}{2}, \quad q=n\sin\theta.$$

Substituting these values

$$3p^2+3p+1-3q^2=3n^2\cos 2\theta+\tfrac{1}{4},$$
$$3q(2p+1)=3n^2\sin 2\theta.$$

Hence, after the double operation on a triangle ABC, we have

$$P_x-Q_x=(3n^2\cos 2\theta+\tfrac{1}{4})(x_1-x_2)-3n^2\sin 2\theta(y_1-y_2),$$
$$P_y-Q_y=(3n^2\cos 2\theta+\tfrac{1}{4})(y_1-y_2)+3n^2\sin 2\theta(x_1-x_2).$$

Hence for any angle of rotation lying between the limits 45° and 135° there will be a value of n, given by

$$n = \pm \frac{1}{2\sqrt{3}} \sqrt{(-\sec 2\theta)},$$

for which the sides of PQR will be at right-angles to those of ABC, and $\frac{1}{4}\tan 2\theta$ times as long. ABC may be a straight line.

In the case of a parallelogram the same substitutions give

$$P_x - Q_x = (2n^2 \cos 2\theta + \tfrac{1}{2})(x_1 - x_2) - 2n^2 \sin 2\theta\,(y_1 - y_2),$$
$$P_y - Q_y = (2n^2 \cos 2\theta + \tfrac{1}{2})(y_1 - y_2) + 2n^2 \sin 2\theta\,(x_1 - x_2).$$

Hence for any angle of rotation lying between the limits 45° and 135° there will be a value of n, given by

$$n = \pm \tfrac{1}{2} \sqrt{(-\sec 2\theta)},$$

for which the sides of $PQRS$ will be at right-angles to those of $ABCD$, and $\frac{1}{2}\tan 2\theta$ times as long. $ABDC$ may be a straight line if $AB = DC$.

Transformation of a triangle into a straight line.

It seems probable that any triangle, by a single (p, q) operation, may be changed into a straight line. This has been worked out for a right-angled triangle.

Let OAB be a triangle right-angled at O. Take O as the origin, OA and OB the x and y axes, and let OA be the point $(a, 0)$, $OB(0, b)$. Then, travelling along OA, and turning anti-clockwise (inwards) we get

$$Z_x = ap + a, \qquad Z_y = aq,$$
$$X_x = -ap - bq, \quad X_y = b + bp - aq,$$
$$Y_x = bq, \qquad Y_y = -bp;$$

XYZ is a straight line if $\dfrac{X_y - Y_y}{X_x - Y_x} = \dfrac{Z_y - X_y}{Z_x - X_x}$.

This condition gives

$$\frac{b + 2bp - aq}{-ap - 2bq} = \frac{-b - bp + 2aq}{a + 2ap + bq} = \frac{b + 3bp}{a - 3bq} = \frac{-b + 3aq}{2a + 3ap},$$

which reduces to

$$ab(3p^2 + 3p + 1) - (a^2 + b^2)q + 3abq^2 = 0.$$

Calling the first term x, we have

$$3abq^3 - (a^2+b^2)\,q + x = 0$$

or
$$q = \frac{a^2+b^2 \pm \sqrt{\{(a^2+b^2)^2 - 12abx\}}}{6ab}.$$

If $x = \frac{ab}{3}$, we get the rational values $q = \frac{a}{3b}$ or $\frac{b}{3a}$ together with $p = -\frac{1}{3}$ or $-\frac{2}{3}$, either value of p being taken with either value of q. The same results are obtained by travelling from A to O, and turning clockwise (inwards).

Hence, if the sides of a right-angled triangle be trisected, and through either point of trisection (taking corresponding points on each side), perpendiculars be drawn inwards equal to $\frac{a}{3b}$ or $\frac{b}{3a}$ times their respective sides, the ends of these perpendiculars are collinear.

Finally, if we travel round the XYZ triangles in the same direction as that by which they were derived, but turn at right-angles in the opposite direction to that used in operating on ABC, we reach points whose coordinates are given by

$$R_x = x_1 + 2p\,(x_1 - x_3) + (p^2+q^2)\,(x_1 + x_2 - 2x_3),$$
$$R_y = y_1 + 2p\,(y_1 - y_3) + (p^2+q^2)\,(y_1 + y_2 - 2y_3).$$

This again shows that a straight line by a double operation may be transformed into a straight line.

Simplest method of obtaining the final figure.

For a triangle put

$$3p^2 + 3p + 1 - 3q^2 = a; \quad 3q\,(2p+1) = b,$$

and let
$$y_1 - y_2 = k; \quad x_1 - x_2 = h,$$

then
$$\frac{P_y - Q_y}{P_x - Q_x} = \frac{ak+bh}{ah-bk} \text{ and } \frac{A_y - B_y}{A_x - B_x} = \frac{k}{h},$$

and so the angle between PQ and AB is

$$\tan^{-1}\frac{\frac{ak+bh}{ah-bk} - \frac{k}{h}}{1 - \frac{ak^2+bhk}{ah^2-bhk}} = \tan^{-1}\frac{b\,(h^2+k^2)}{a\,(h^2+k^2)} = \tan^{-1}\frac{b}{a},$$

so that the angle between any pair of corresponding sides of PQR and ABC is the same.

Again

$$PQ = \sqrt{\{(ak+bh)^2 + (ah-bk)^2\}}$$
$$= \sqrt{\{(a^2+b^2)(h^2+k^2)\}} = AB\sqrt{(a^2+b^2)}.$$

Hence, for all values of p and q, PQR is similar to ABC, and may be obtained by magnifying the lines from the centroid to the corners of $ABC\sqrt{(a^2+b^2)}$ times, and then rotating about the centroid through $\tan^{-1}\frac{b}{a}$. ABC may be a straight line.

Similar statements hold for parallelograms if we put $a = 2p^2+2p+1-2q^2$ and $b = 2q(2p+1)$.

If $p = -\frac{1}{2}$, $q = \pm\frac{\sqrt{5}}{2\sqrt{3}}$ in the case of a triangle, $b = 0$ and $a = -1$, so that PQR is obtained by simply rotating ABC about its centroid through two right-angles. For a parallelogram, the corresponding result requires $p = -\frac{1}{2}$, $q = \pm\frac{\sqrt{3}}{2}$, a case which has been previously noticed.

The general case.

If we perform the (p, q) operation on the straight lines joining the point which divides AB in the ratio $1:k$ to the point dividing BC in the ratio $1:l$, and similarly for each pair of sides, we get similar results and similar special cases to those obtained by operating on the sides of ABC and XYZ, though the values of p and q required to obtain these special cases are more complicated than in the latter problem. The values, k or l equals infinity, $k = l$, or $k = \frac{1}{l}$, lead to a considerable simplification.

In the general case, PQR may be made to coincide with ABC. As a simple example, if we operate on lines joining one corner of a triangle to a point in the opposite side dividing it in the ratio of $1:l$, the triangles ABC and XYZ reproduce each other for $p = \frac{l}{(l^2+l+1)}$, $q = 0$.

PROPERTIES OF A PAIR OF ORTHOGONANTS.

By *Thomas Muir*, *LL.D.*

1. THERE is more than one direction in which it has been proposed to extend the term *orthogonant*. Sylvester's suggestion* was that a determinant should be so called *if the ratio of any one of its elements to the corresponding element of the adjugate were constant*. Another proposal was that the name should be given to any determinant *which has the product of every pair of its rows equal to* 0 *and the square of every row equal to a constant*. The two extensions are not essentially different. The latter, however, is perhaps the more natural, and at any rate it has met with more acceptance. In adopting it we shall need to speak of any determinant which fulfills the conditions as being an orthogonant whose *base* is the constant in question. The ordinary orthogonant has 1 for its base.

2. Known theorems regarding the ordinary orthogonant are of course readily extensible to orthogonants whose base is σ. The following are worth specially noting.

(*a*) *The product of two equigrade orthogonants is an orthogonant whose base is the product of the two bases.*

(*b*) *The sum of the squares of the k-line minors formable from any array of k rows of an orthogonant, whose base is* σ, *is* σ^k.

(*c*) *The co-factor of any k-line minor of an n-line orthogonant, whose base is* σ, *is equal to the said minor multiplied by* $\sigma^{\frac{1}{2}n-k}$.

(*d*) *The adjugate of an n-line orthogonant, whose base is* σ, *is an orthogonant whose base is* σ^{n-1}.

3. Let us now take any two orthogonants Ω_1, Ω_2 of the n^{th} order and with bases σ_1, σ_2 respectively, and let us denote the columns of the former by

$$1,\ 2,\ \ldots,\ n,$$

and those of the latter by

$$1',\ 2',\ \ldots,\ n';$$

* Sylvester, J. J., "Thoughts on orthogonal matrices . . . ," *Phil. Magazine*, vol. xxxiv. (1867), pp. 461–475.

so that there may be ready to hand a convenient notation for a determinant composed of k-columns taken from the one and $n-k$ columns taken from the other. For example, n being 4, he determinant, whose first and fourth columns are the first and fourth columns of Ω_1, and whose second and third columns are the second and fourth of Ω_2, is denoted by

$$|12'4'4| \quad \text{or} \quad 12'4'4,$$

and if Ω_1 be $|\alpha_1\beta_2\gamma_3\delta_4|$ and Ω_2 be $|\epsilon_1\zeta_2\eta_3\theta_4|$, then

$$12'4'4 \equiv \begin{vmatrix} \alpha_1 & \epsilon_2 & \epsilon_4 & \alpha_4 \\ \beta_1 & \zeta_2 & \zeta_4 & \beta_4 \\ \gamma_1 & \eta_2 & \eta_4 & \eta_4 \\ \delta_1 & \theta_2 & \theta_4 & \delta_4 \end{vmatrix}.$$

4. *The determinant which has for its first $n-1$ columns the corresponding $n-1$ columns of Ω_1 and for its remaining column the p^{th} column of Ω_2 is equal to*

$$\sigma_1^{\frac{1}{2}n-1}(\text{col}_{p'}.\text{col}_n) \ldots\ldots\ldots\ldots\ldots\ldots (e)$$

or, say,

$$\sigma_1^{\frac{1}{2}n-1}(p'.n).$$

Taking $n=4$, $p=3$, and Ω_1, Ω_2 as in the preceding paragraph we have

$$123' \equiv \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \epsilon_3 \\ \beta_1 & \beta_2 & \beta_3 & \zeta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 & \eta_3 \\ \delta_1 & \delta_2 & \delta_3 & \theta_3 \end{vmatrix},$$

and multiplying the left-hand side by σ_1^2 and the right-hand side columnwise by $|\alpha_1\beta_2\gamma_3\delta_4|$ we obtain

$$1233'.\sigma_1^2 = \begin{vmatrix} \sigma_1 & . & . & . \\ . & \sigma_1 & . & . \\ . & . & \sigma_1 & . \\ 3'.1 & 3'.2 & 3'.3 & 3'.4 \end{vmatrix} = \sigma_1^3(3'.4),$$

and therefore

$$1233' = \sigma_1(3'.4).$$

5. From this we readily pass to an extension of it, namely, *the determinant which has for $n-1$ of its columns the corre-*

sponding $n-1$ columns of Ω_1, the omitted column being the n^{th}, and for its remaining column the p^{th} column of Ω_2 is equal to

$$\sigma_1^{\frac{1}{2}n-1}(p'.h) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (f).$$

As it is exactly the same thing to say that if the p^{th} column of Ω_2 be replaced by the p^{th} column of Ω_1, the resulting determinant is equal to

$$\sigma_2^{\frac{1}{2}n-1}(h.p'),$$

we are enabled to conclude that *if Ω_1, Ω_2 be altered by the interchange of the n^{th} column of the former with the p^{th} column of the latter, the ratio of the resulting determinants is*

$$\left(\frac{\sigma_1}{\sigma_2}\right)^{\frac{1}{2}n-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (g).$$

For example, we have from § 3

$$1233' = \sigma_1(3'.4),$$

and from § 5

$$1'2'44' = \sigma_2(4.3'),$$

and therefore

$$1233' : 1'2'44' = \sigma_1 : \sigma_2.$$

6. *If any column of Ω_1 be multiplied by each column of Ω_2, the sum of the squares of the resulting products is $\sigma_1\sigma_2$.* (*h*).

Taking the Ω's of the fourth order as above, and choosing the fourth column of Ω_1, we have

$$\begin{aligned}
4.1' &= \alpha_4\epsilon_1 + \beta_4\zeta_1 + \gamma_4\eta_1 + \delta_4\theta_1,\\
4.2' &= \alpha_4\epsilon_2 + \beta_4\zeta_2 + \gamma_4\eta_2 + \delta_4\theta_2,\\
4.3' &= \alpha_4\epsilon_3 + \beta_4\zeta_3 + \gamma_4\eta_3 + \delta_4\theta_3,\\
4.4' &= \alpha_4\epsilon_4 + \beta_4\zeta_4 + \gamma_4\eta_4 + \delta_4\theta_4,
\end{aligned}$$

and on squaring the right-hand members and performing addition the co-factor of α_4^2 is seen to be

$$\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \epsilon_4^2, \quad i.e., \quad \sigma_2,$$

and the co-factor of $2\alpha_4\beta_4$ to be

$$\epsilon_1\zeta_1 + \epsilon_2\zeta_2 + \epsilon_3\zeta_3 + \epsilon_4\zeta_4, \quad i.e., \quad 0.$$

Attention to the other similar co-factors gives us as the total

$$(\alpha_4^2 + \beta_4^2 + \gamma_4^2 + \delta_4^2)\,\sigma_2, \quad i.e., \quad \sigma_1\sigma_2,$$

as required.

7. *The sum of the squares of the n determinants formed from Ω_1 by interchanging its n^{th} column with all the columns of Ω_2 in succession is equal to $\sigma_1^{n-1}\sigma_2$.* (*i*).

For the sum in question is by (f) equal to

$$\sigma_1^{n-2}\{(1'.h)^2+(2'.h)^2+\ldots+(n'.n)^2\},$$

and therefore from (h) is equal to

$$\sigma_1^{n-2}.\sigma_1\sigma_2, \quad i.e., \quad \sigma_1^{n-1}\sigma_2,$$

There is, however, another mode of proof which, though not shorter, is more interesting, the result $\sigma_1^{n-1}\sigma_2$ coming out at first not in the form $\sigma_1^{n-2}.\sigma_1\sigma_2$, but in the form $\sigma_1^{n-1}.\sigma_2$. Thus, returning to $|\alpha_1\beta_2\gamma_3\delta_4|$ and denoting the co-factors of the elements of its last column by A_4, B_4, C_4, D_4, we clearly have

$$\begin{aligned}|1231'|^2+|1232'|^2+|1233'|^2+|1234'|^2 \\ = \quad (\epsilon_1A_4+\zeta_1B_4+\eta_1C_4+\theta_1D_4)^2 \\ +(\epsilon_2A_4+\zeta_2B_4+\eta_2C_4+\theta_2D_4)^2 \\ +(\epsilon_3A_4+\zeta_2B_4+\eta_3C_4+\theta_3D_4)^2 \\ +(\epsilon_4A_4+\zeta_4B_4+\eta_4C_4+\theta_4D_4)^2.\end{aligned}$$

Squaring and adding as in § 6 we obtain

$$(A_4^2+B_4^2+C_4^2+D_4^2)\,\sigma_2,$$

and therefore by § 2 (d),

$$\sigma_1^3.\sigma_2.$$

8. *The determinant which has for its first $n-2$ columns the corresponding $n-2$ columns of Ω_1 and for its remaining pair of columns the p^{th} and q^{th} of Ω_2 is equal to*

$$\sigma_1^{\frac{1}{2}n-2}\left\|\begin{matrix}\text{col}_{p'}\\ \text{col}_{q'}\end{matrix}\right\|.\left\|\begin{matrix}\text{col}_{n-1}\\ \text{col}_{n}\end{matrix}\right\| \quad \ldots\ldots\ldots\ldots\ldots(j).$$

Taking $n=4$, $p=1$, $q=2$ we have

$$121'2'.\sigma_1^2=\begin{vmatrix}\alpha_1 & \alpha_2 & \epsilon_1 & \epsilon_2\\ \beta_1 & \beta_2 & \zeta_1 & \zeta_2\\ \gamma_1 & \gamma_2 & \eta_1 & \eta_2\\ \delta_1 & \delta_2 & \theta_1 & \theta_2\end{vmatrix}.\begin{vmatrix}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\\ \beta_1 & \beta_2 & \beta_3 & \beta_4\\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4\\ \delta_1 & \delta_2 & \delta_3 & \delta_4\end{vmatrix}$$

$$=\begin{vmatrix}\sigma_1 & . & . & .\\ . & \sigma_1 & . & .\\ 1'.1 & 1'.2 & 1'.3 & 1'.4\\ 2'.1 & 2'.2 & 2'.3 & 2'.4\end{vmatrix},$$

and therefore

$$121'2' = \begin{vmatrix} 1'.3 & 1'.4 \\ 2'.3 & 2'.4 \end{vmatrix} = \begin{Vmatrix} \epsilon_1 & \zeta_1 & \eta_1 & \theta_1 \\ \epsilon_2 & \zeta_2 & \eta_2 & \theta_2 \end{Vmatrix} \cdot \begin{Vmatrix} \alpha_3 & \beta_3 & \gamma_3 & \delta_3 \\ \alpha_4 & \beta_4 & \gamma_4 & \delta_4 \end{Vmatrix},$$

as the theorem to be proved requires.

From this we can readily proceed to the extension (k) corresponding to the extension which (f) is of (e): also to the proposition corresponding to (g), namely, that *if* Ω_1, Ω_2 *be altered by the interchange of the* n^{th} *and* k^{th} *columns of the former with the* p^{th} *and* q^{th} *columns of the latter, the ratio of the resulting determinants is*

$$\left(\frac{\sigma_1}{\sigma_2}\right)^{\frac{1}{2}n-2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (l).$$

9. *If any pair of columns of* Ω_1 *be multiplied by each pair of columns of* Ω_2, *the sum of the squares of the resulting products is equal to* $\sigma_1^2\sigma_2^2$. (m)

Taking $|\alpha_1\beta_2\gamma_3\delta_4|$, $|\epsilon_1\zeta_2\eta_3\theta_4|$ for Ω_1, Ω_2 as above, and 3, 4 as the fixed pair of columns of Ω_1, we have for our sum of squares of products

$$\begin{aligned} &\left\{\begin{Vmatrix} \alpha_3\beta_3\gamma_3\delta_3 \\ \alpha_4\beta_4\gamma_4\delta_4 \end{Vmatrix} \cdot \begin{Vmatrix} \epsilon_1\zeta_1\eta_1\theta_1 \\ \epsilon_2\zeta_2\eta_2\theta_2 \end{Vmatrix}\right\}^2 \\ &+ \left\{\begin{Vmatrix} \alpha_3\beta_3\gamma_3\delta_3 \\ \alpha_4\beta_4\gamma_4\delta_4 \end{Vmatrix} \cdot \begin{Vmatrix} \epsilon_1\zeta_1\eta_1\theta_1 \\ \epsilon_3\zeta_3\eta_3\theta_3 \end{Vmatrix}\right\}^2 \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ \left\{\begin{Vmatrix} \alpha_3\beta_3\gamma_3\delta_3 \\ \alpha_4\beta_4\gamma_4\delta_4 \end{Vmatrix} \cdot \begin{Vmatrix} \epsilon_3\zeta_3\eta_3\theta_3 \\ \epsilon_4\zeta_4\eta_4\theta_4 \end{Vmatrix}\right\}^2. \end{aligned}$$

Each product is expressible as the sum of six products of pairs of two-line determinants, so that this becomes

$$\begin{aligned} &\left\{|\alpha_3\beta_4|\cdot|\epsilon_1\zeta_2| + |\alpha_3\gamma_4|\cdot|\epsilon_1\eta_2| + \ldots + |\gamma_3\delta_4|\cdot|\eta_1\theta_2|\right\}^2 \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ \left\{|\alpha_3\beta_4|\cdot|\epsilon_3\gamma_4| + |\alpha_3\zeta_4|\cdot|\epsilon_3\eta_4| + \ldots + |\gamma_3\delta_4|\cdot|\eta_3\theta_4|\right\}^2. \end{aligned}$$

On squaring and collecting terms that have a common factor it is seen that the complete co-factor of $|\alpha_3\beta_4|^2$ is

$$|\epsilon_1\zeta_2|^2 + |\epsilon_1\zeta_3|^2 + \ldots + |\epsilon_3\zeta_4|^2$$

or

$$\begin{Vmatrix} \epsilon_1\epsilon_2\epsilon_3\epsilon_4 \\ \zeta_1\zeta_2\zeta_3\zeta_4 \end{Vmatrix}^2,$$

and therefore by (b) is equal to

$$\sigma_2^{\,2};$$

also that the complete co-factor of $2\,|\alpha_3\beta_4|\,.\,|\alpha_3\gamma_4|$ is

$$|\epsilon_1\zeta_2|\,.\,|\epsilon_1\eta_2|+|\epsilon_1\zeta_3|\,.\,|\epsilon_1\eta_3|+\ldots+|\epsilon_3\zeta_4|\,.\,|\epsilon_3\eta_4|$$

or

$$\left\|\begin{matrix}\epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4\\ \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4\end{matrix}\right\|\,.\,\left\|\begin{matrix}\epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4\\ \eta_1 & \eta_2 & \eta_3 & \eta_4\end{matrix}\right\|,$$

and is therefore equal to

$$\left|\begin{matrix}\sigma_1 & \cdot\\ \cdot & \cdot\end{matrix}\right|, \quad i.e., \quad 0.$$

The like consideration of the other similar co-factors gives us finally the total

$$\left\{|\alpha_3\beta_4|^2+|\alpha_3\gamma_4|^2+\ldots+|\gamma_3\delta_4|^2\right\}\sigma_2^{\,2},$$

which by (b) is equal to

$$\sigma_1^{\,2}\sigma_2^{\,2}.$$

10. *The sum of the squares of the* $\frac{1}{2}n\,(n-1)$ *determinants formed from* Ω_1 *by interchanging a fixed pair of its columns with every possible pair of the columns of* Ω_2 *is* $\sigma_1^{\,n-2}\sigma_2^{\,2}$. (n)

For, if the fixed columns of Ω_1 be the h^{th} and k^{th}, the sum in question is by (j) equal to

$$\begin{aligned}
&\sigma_1^{\,n-4}\left\{\left\|\begin{matrix}\text{col}1'\\ \text{col}2'\end{matrix}\right\|\,.\,\left\|\begin{matrix}\text{col}_h\\ \text{col}_k\end{matrix}\right\|\right\}^2\\
&+\sigma_1^{\,n-4}\left\{\left\|\begin{matrix}\text{col}1'\\ \text{col}3'\end{matrix}\right\|\,.\,\left\|\begin{matrix}\text{col}_h\\ \text{col}_k\end{matrix}\right\|\right\}^2\\
&+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&+\sigma_1^{\,n-4}\left\{\left\|\begin{matrix}\text{col}(n-1)'\\ \text{col}n'\end{matrix}\right\|\,.\,\left\|\begin{matrix}\text{col}_h\\ \text{col}_k\end{matrix}\right\|\right\}^2,
\end{aligned}$$

and therefore from (m) is equal to

$$\sigma^{n-4}.\sigma_1^{\,2}\sigma_2^{\,2}, \quad i.e., \quad \sigma_1^{\,n-2}\sigma_2^{\,2},$$

11. Pausing now to look back on what precedes it will be seen that, just as the propositions (e), (f), (g), (h), (i) concern the transference of one column of an orthogonant, the propositions (j), (k), (l), (m), (n) concern the transference of two.

Further, it will be felt that there must be five similar results when the number of transformed columns is r, and that they can be readily formulated on the same model. This done, the following interesting theorem is reached.

12. *If there be two n-line orthogonants Ω_1, Ω_2 whose bases are σ_1, σ_2, and from their $2n$ columns every possible set of n columns be taken, the sum of the squares of the determinants so formable is equal to*

$$(\sigma_1 + \sigma_2)^n \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(o).$$

The number of determinants in question

$$C_{2n,n}$$

may be partitioned into

$$1^2 + n^2 + \left\{\frac{n(n-1)}{1.2}\right\}^2 + \ldots + n^2 + 1^2,$$

the 1^2 corresponding to Ω_1, the n^2 to those formed from Ω_1 by leaving out one column and putting in place of it each column of Ω_2, the $\{\frac{1}{2}n(n-1)\}^2$ to those formed from Ω_1 by leaving out two columns and putting in place of them every possible pair obtainable from Ω_2, and so on. But then we know that

$$\Omega_1^{\,2} = \sigma_1^{\,n};$$

that the sum of the squares of those obtained from Ω_1 by the transference of a single column is

$$n \cdot \sigma_1^{\,n-1}\sigma_2;$$

that the sum of the squares of those obtained by the similar transference of a pair of columns is

$$\frac{n(n+1)}{1.2} \cdot \sigma_1^{\,n-2}\sigma_2^{\,2};$$

and so on. The grand total thus is

$$\sigma_1^{\,n} + n \cdot \sigma_1^{\,n-1}\sigma_2 + \frac{n(n-1)}{1.2} \cdot \sigma_1^{\,n-2}\sigma_2^{\,2} + \ldots,$$

i.e.,

$$(\sigma_1 + \sigma_2)^n,$$

as asserted.

Capetown, S.A.
September 12th, 1914.

NOTE ON A CERTAIN PARTIAL DIFFERENTIAL EQUATION OF THE SECOND ORDER POSSESSING AN INTERMEDIATE INTEGRAL OF THE FIRST ORDER.

By *J. R. Wilton, M.A., D.Sc.*

It appears not to have been remarked that the equation

$$\frac{rt-s^2}{pt-qs} = F\left(\frac{qr-ps}{pt-qs},\quad \frac{q^2r-2pqs+p^2t}{pt-qs},\ z\right).....(1),$$

in which F is an arbitrary functional form, always possesses an intermediate integral of the first order, which may very easily be obtained as follows :—

We assume that

$$\frac{qr-ps}{pt-qs} = a,$$

where a is constant. Then equation (1) becomes

$$\frac{r+as}{p} \equiv \frac{s+at}{q} = F(a,\ p+aq,\ z),.....(2).$$

Let $p+aq=u$, and determine a function ϕ, such that

$$F(a,\ u,\ z) = -\frac{\partial\phi}{\partial z} \Big/ \frac{\partial\phi}{\partial u}.$$

This requires the integration of the ordinary differential equation

$$\frac{du}{dz} = F(a,\ u,\ z).....(3),$$

and ϕ may be regarded as the arbitrary constant which occurs in the solution of this equation.

When ϕ has been determined from (3) we substitute in (2) and obtain the two equations

$$(r+as)\frac{\partial\phi}{\partial u} + p\frac{\partial\phi}{\partial z} = 0,$$

$$(s+at)\frac{\partial\phi}{\partial u} + q\frac{\partial\phi}{\partial z} = 0,$$

the integral of which is evidently

$$\phi(p+aq, z, a)=b.$$

This is a complete integral; the general integral is obtained by eliminating a between

$$\phi(p+aq, z, a)=\psi(a),$$

and

$$q\frac{\partial\phi}{\partial u}+\frac{\partial\phi}{\partial a}=\psi'(a),$$

in which ψ is an arbitrary function of a.

As examples let us consider the two equations suggested by Prof. Forsyth in his lecture* before the International Congress of mathematicians at Rome, namely

$$(rt-s^2)(sp-rq)-(sq-tp)^2=0.....(5),$$

and

$$z(rt-s^2)^2-(tp^2-2spq+rq^2)(rt-s^2)+(tp-sq)(sp-rq)=0...(6).$$

Equation (5) may be written

$$\frac{rt-s^2}{pt-qs}+\frac{pt-qs}{qr-ps}=0,$$

which leads to $$r+as+p/a=0,$$

and to $$s+at+q/a=0,$$

whence $$p+aq+z/a=b/a,$$

or $$z+ap+a^2q=b,$$

is the complete intermediate integral of (5).

Again, equation (6) may be written

$$z\left(\frac{rt-s^2}{pt-qs}\right)^2-\frac{rt-s^2}{pt-qs}\cdot\frac{q^2r-2pqs+p^2t}{pt-qs}-\frac{qr-ps}{pt-qs}=0,$$

which leads to

$$z\left(\frac{r+as}{p}\right)^2-\frac{r+as}{p}(p+aq)-a=0.....(7),$$

---

* "On the present condition of partial differential equations of the second order, as regards formal integration." *Atti del* iv° *Congresso Internazionale dei Matematici* (Roma, 6–11 Aprile 1908). See also his *Theory of Differential Equations*, Vol. vi, p. 222, Ex. 1, and p. 225, Ex. 3.

and to

$$z\left(\frac{s+at}{q}\right)^2-\frac{s+at}{q}(p+aq)-a=0\ldots\ldots\ldots(8).$$

From (7) we find

$$\frac{r+as}{p}=\frac{\sqrt{\{(p+aq)^2+4az\}}+p+aq}{2z}\ldots\ldots\ldots(9).$$

Equation (3) here becomes

$$2z\frac{du}{dz}=(u^2+4az)^{\frac{1}{2}}+u,$$

an integral of which is

$$(u^2+4az)^{\frac{1}{2}}-u=\phi.$$

We therefore re-write (9) in the form

$$(r+as)\frac{p+aq-\{(p+aq)^2+4az\}^{\frac{1}{2}}}{\{(p+aq)^2+4az\}^{\frac{1}{2}}}+\frac{2ap}{\{(p+aq)^2+4az\}^{\frac{1}{2}}}=0,$$

from which, and the corresponding equation deduced from (8), we find

$$\{(p+aq)^2+4az\}^{\frac{1}{2}}-(p+aq)=2b,$$

leading to the complete integral

$$az=b^2+b(p+aq),$$

which may be written

$$z+hp+kq=hk,$$

h and k being constants.

It is evidently possible somewhat to extend the form of (1), but such extensions are of slight interest in comparison with (1) itself.

An evident generalisation consists in the determination of those equations which have a first order integral of the form

$$f(u+av,\ a,\ z)=b,$$

in which f is any given function, and u and v are given functions of x, y, z, p and q, and in the integration of the resulting equation.

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NOTE ON THE PRODUCT FOR $\sin x$.

By *O. A. Corte.*

§ 1. In *The Quarterly Journal of Mathematics*, vol. xvii. (1881), p. 211, Dr. Glaisher proved the identities:

$$\cot x = \frac{1}{x} + \frac{1}{x-\pi} + \frac{1}{x+\pi} + \frac{1}{x-2\pi} + \frac{1}{x+2\pi} + \ldots,$$

$$\operatorname{cosec} x = \frac{1}{x} - \frac{1}{x-\pi} - \frac{1}{x+\pi} + \frac{1}{x-2\pi} + \frac{1}{x+2\pi} - \ldots,$$

as limits of purely algebraical theorems by starting from factor expressions which on becoming infinite give $\cot x$ and $\operatorname{cosec} x$.

The following shows how the more complex identity:

$$\frac{1}{x^2(1-x^3/1^3)^2(1-x^3/2^3)^2\cdots}$$

$$= \frac{1}{x^2} + \frac{1}{(x-1)^2} + \frac{1}{(x+1)^2} + \frac{1}{(x-2)^2} + \frac{1}{(x+2)^2} + \ldots (1),$$

may also be proved directly, the details of the proof in stepping from the finite to the transcendental equation—which are here more troublesome than in Dr. Glaisher's case—being given.

The result seems of importance, for by its aid we can readily identify the infinite product

$$x\left(1-\frac{x^2}{1^2}\right)\left(1-\frac{x^2}{2^2}\right)\ldots,$$

with its value $\frac{1}{\pi}\sin \pi x$,

as will be shown in a corollary.

§ 2. Let $G(x) = x\left(1-\frac{x^2}{1^2}\right)\left(1-\frac{x^2}{2^2}\right)\ldots,$

and $R_n(x) = \left\{1-\frac{x^2}{(n+1)^2}\right\}^2\left\{1-\frac{x^2}{(n+2)^2}\right\}^2\ldots;$

then we know that if m is an integer (see Hobson, *Plane Trigonometry*, § 290):

$$G(m-y) = (-1)^{m+1}\, G(y) \ldots\ldots\ldots\ldots\ldots (2).$$

Now, by the rule for partial fractions,

$$\frac{R_n(x)}{\{G(x)\}^2} = \frac{1}{x^2(1-x^2/1^2)^2 \dots (1-x^2/n^2)^2}$$

$$= \frac{A}{x^2} + \sum_{m=1}^{n} \frac{A_m}{(1-x^2/m^2)^2} + \sum_{m=1}^{n} \frac{B_m}{(1-x^2/m^2)} \dots (3).$$

Here $A=1$; $A_m = \lim_{x\to m} \left\{ \left(1-\frac{x^2}{m^2}\right)^2 \frac{R_n(x)}{\{G(x)\}^2} \right\}$

$$= \lim_{y\to 0} \left\{ \left(\frac{2m-y}{m^2}\right)^2 \frac{y^2}{\{G(y)\}^2} R_n(m-y) \right\},$$

putting $x=m-y$ and using (2); whence

$$A_m = \frac{4R_n(m)}{m^2} \dots (4).$$

On the other hand

$$B_m = \lim_{x\to m} \left\{ \left(1-\frac{x^2}{m^2}\right) \left[\frac{R_n(x)}{\{G(x)\}^2} - \frac{A_m}{(1-x^2/m^2)^2} \right] \right\}$$

$$= \lim_{x\to m} \left\{ \frac{1}{(1-x^2/m^2)} \left[R_n(x) \frac{(1-x^2/m^2)^2}{\{G(x)\}^2} - \frac{4R_n(m)}{m^2} \right] \right\},$$

or, putting $x=m-y$,

$$B_m = \lim_{y\to 0} \left\{ \frac{1}{2.y/m} \left[\{R_n(m) - yR'_n(m)\} \frac{y^2}{\{G(y)\}^2} \right.\right.$$

$$\left.\left. \times \frac{(2m-y)^2}{m^4} - \frac{4R_n(m)}{m^2} \right] \right\},$$

whence $$B_m = -\frac{2R_n(m)}{m^2} - \frac{2R'_n(m)}{m} \dots (5).$$

Hence, from (4) and (5), (3) becomes

$$\frac{R_n(x)}{\{G(x)\}^2} = \frac{1}{x^2} + \sum_{m=1}^{n} \frac{2(m^2+x^2)R_n(m)}{(m^2-x^2)^2} - 2\sum_{m=1}^{n} \frac{mR'_n(m)}{m^2-x^2} \dots (6).$$

§3. Now

$$\left| \frac{2(m^2+x^2)}{(m^2-x^2)^2} R_n(m) \right| < \left| \frac{2(m^2+x^2)}{(m^2-x^2)^2} \right| < u_m, \text{ (say)};$$

also $$\lim_{n\to\infty} R_n(m) = 1,$$

when m is fixed, and $\sum_m u_m$ is generally convergent; therefore by Tannery's theorem (see Bromwich, *Infinite Series*, § 49):

$$\lim_{n\to\infty} \sum_{m=1}^{n} \frac{2(m^2+x^2)R_n(m)}{(m^2-x^2)^2} = \sum_{m=1}^{\infty} \frac{2(m^2+x^2)}{(m^2-x^2)^2} \ldots\ldots (7).$$

Again, consider $\sum_{m=1}^{n} \frac{mR'_n(m)}{m^2-x^2}$;

differentiating $R_n(m)$ logarithmically:

$$R'_n(m) = -4R_n(m) \sum_{s=n+1}^{\infty} \left(\frac{m}{s^2-m^2}\right) \ldots\ldots\ldots\ldots (8),$$

the process being justified by the result being uniformly convergent in the m-range $(1, \nu)$.

If m be fixed, $R'_n(m)$ has then the limit zero and thus we can write, taking the limiting sign through the first $(\nu-1)$ terms,

$$\lim_{n\to\infty} \sum_{m=1}^{n} \left\{\frac{mR'_n(m)}{m^2-x^2}\right\} = \lim_{n\to\infty} \sum_{m=\nu}^{n} \left\{\frac{mR'_n(m)}{m^2-x^2}\right\} \ldots\ldots\ldots (9),$$

ν being an integer at our disposal, which we shall suppose greater than $|x|$.

Let us substitute for $R_n(m)$ in (8); we can then write:

$$\frac{mR'_n(m)}{x^2-m^2} = \frac{4}{(1-x^4/m^2)} \times \sum_{s=n+1}^{\infty} \left\{\frac{1}{s^2}\left[\left(1-\frac{m^2}{(n+1)^2}\right)^2 \left(1-\frac{m^2}{(n+2)^2}\right)^3 \ldots \left(1-\frac{m^2}{s^2}\right)\ldots\right]\right\},$$

and therefore, since $n+1 > m > \nu > |x|$, $\frac{mR'_n(m)}{x^2-m^2}$ is positive and diminishes as m increases, x being supposed real*. Hence by the Cauchy-Maclaurin theorem:

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* Mr. Watson has offered me a proof which dispenses with this restriction on x and appeals to Tannery's theorem for infinite integrals, the arguments in the text being practically those by which Tannery's theorem is proved.

"If A be the upper bound of $\left|1-\frac{x^2}{m^2}\right|^{-1}$,

then
$$\left|\sum_{m=1}^{n} \frac{mR'_n(m)}{m^2-x^2}\right| < A \sum_{m=1}^{n} \left\{-\frac{R'_n(m)}{m}\right\}$$
$$< A\left\{-R'_n(1) - \int_1^n \frac{R'_n(y)}{y}\,dy\right\}$$
$$< A\left\{-R'_n(1) - \left[\frac{R_n(y)}{y}\right]_1^n - \int_1^n \frac{R_n(y)}{y^2}\,dy\right\},$$

which gives zero limit by using Tannery's theorem for infinite integrals (se Bromwich, *Infinite Series*, § 172)."

$$\lim_{n\to\infty} \sum_{m=\nu}^{n} \left\{\frac{mR'_n(m)}{x^2-m^2}\right\} \leq \lim_{n\to\infty} \left\{\int_\nu^n \frac{zR'_n(z)}{x^2-z^2}\,dz + \frac{\nu R'_n(\nu)}{\nu^2-x^2}\right\}$$

$$i.e. \leq \lim_{n\to\infty} \int_\nu^n \frac{zR'_n(z)}{x^2-z^2}\,dz \ldots\ldots\ldots\ldots(10);$$

but $$\int_\nu^n \frac{zR'_n(z)}{x^2-z^2}\,dz = \left[\frac{zR_n(z)}{x^2-z^2}\right]_\nu^n - \int_\nu^n R_n(z)\,\frac{d}{dz}\left(\frac{z}{x^2-z^2}\right)dz,$$

and therefore, dropping the last integral which is positive,

$$\int_\nu^n \frac{zR'_n(z)\,dz}{x^2-z^2} < \frac{\nu R_n(\nu)}{\nu^2-x^2} - \frac{nR_n(n)}{n^2-x^2} < \frac{\nu}{\nu^2-x^2}.$$

But ν is at our disposal and by taking it sufficiently large we can make

$$\int_\nu^n \frac{zR'_n(z)\,dz}{x^2-z^2}$$

as small as we please, whatever n; hence, from (9) and (10),

$$\lim_{n\to\infty} \sum_{m=1}^{n} \left\{\frac{mR'_n(m)}{m^2-x^2}\right\} = \lim_{n\to\infty} \sum_{m=\nu}^{n} \left\{\frac{mR'_n(m)}{m^2-x^2}\right\} = 0 \ldots(11).$$

(7) and (11) give us the limiting form of (6), *i.e.*

$$\frac{1}{\{G(x)\}^2} = \frac{1}{x^2} + \sum_{m=1}^{\infty} \frac{2(m^2+x^2)}{(m^2-x^2)^2} \ldots\ldots\ldots\ldots(12),$$

or

$$\frac{1}{x^2(1-x^2/1^2)^2 \ldots (1-x^2/m^2)^2 \ldots} = \frac{1}{x^2} + \sum_{m=1}^{\infty} \left\{\frac{1}{(m-x)^2} + \frac{1}{(m+x)^2}\right\},$$

as we wanted to show.

§ 4. *Corollary.* Putting $y = G(x)$ and integrating (12) twice we find that

$$\int\int \frac{dx}{y^2} = -\log y,$$

the series being uniformly convergent so long as x is not an integer. Thus $G(x)$ satisfies the differential equation

$$\frac{1}{y^2} + \frac{d^2}{dx^2}(\log y) = 0,$$

i.e. $$1 + y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = 0,$$

whose general solution is $y = c \sin \frac{x+a}{c}$.

Since y is an odd function of x, $a = 0$; also

$$G(x) = (-1)^{r+1}\, G(r-x),$$

while $$c \sin \frac{x}{c} = (-1)^{r+1}\, c \sin\left(\frac{rc\pi - x}{c}\right);$$

it follows that $c = \frac{1}{\pi}$ and

$$x\left(1 - \frac{x^2}{1^2}\right)\left(1 - \frac{x^2}{2^2}\right)\ldots = \frac{1}{\pi}\sin \pi x.$$

My thanks are due to Mr. J. H. Dibb, who originally checked the work, and to Mr. G. N. Watson, who has favoured me with the footnote in which he is referred to, and has kindly helped me to condense the manuscript.

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NOTE ON AN ELIMINATION QUOTED BY CAYLEY.

By *Prof. J. E. A. Steggall.*

In the *Messenger of Mathematics*, vol. xi., pp. 23–25*, Cayley gives a note on the following problem:—

"Prove that if $a+b+c=0$ and $x+y+z=0$, then

$$\begin{aligned} &4(ax+by+cz)^3 \\ &\quad -3(ax+by+cz)(a^2+b^2+c^2)(x^2+y^2+z^2) \\ &\quad -2(b-c)(c-a)(a-b)(y-z)(z-x)(x-y) \\ &\quad -54abcxyz = 0." \end{aligned}$$

He gives an elegant, but as he says, "not very simple way" of proving it by determinants: I venture to write out the more direct proof which has just occurred to me in re-reading his note.

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* "Solution of a Senate House Problem." The paper is reprinted in Cayley's *Collected Mathematical Papers*, vol. xi., pp. 265–267.

Consider the equation of which the roots are

$$ax+by+cz, \quad bx+cy+az, \quad cx+ay+bz:$$

their sum is clearly 0, and that of their products two at a time is given by

$$s_2=(x^2+y^2+z^2)(bc+ca+ab)$$
$$+(yz+zx+zy)(a^2+b^2+c^2+bc+ca+ab),$$

which, by virtue of the given relations, is

$$-\tfrac{3}{4}(a^2+b^2+c^2)(x^2+y^2+z^2).$$

The product is

$$(y^2z+z^2x+x^2y)(b^2c+c^2a+a^2b)$$
$$+(yz^2+zx^2+xy^2)(bc^2+ca^2+ab^2)$$
$$+(x^3+y^3+z^3)abc+3abcxyz$$
$$+(a^3+b^3+c^3)xyz.$$

Now

$$b^2c+c^2a+a^2b+bc^2+ca^2+ab^2\equiv A=-3abc,$$

$$b^2c+c^2a+a^2b-bc^2-ca^2-ab^2\equiv B=-(b-c)(c-a)(a-b),$$

and similarly for expressions in x, y, z.

Hence, using also

$$x^3+y^3+z^3=3xyz,$$

the product is

$$\tfrac{1}{2}(A+B).\tfrac{1}{2}(X+Y)+\tfrac{1}{2}(A-B).\tfrac{1}{2}(X-Y)+9abcxyz$$
$$=\tfrac{1}{2}(AX+BY)+9abcxyz=\tfrac{27}{2}abcxyz+\tfrac{1}{2}(BY),$$

Hence $ax+by+cz$ is a root of the equation

$$t^3-\tfrac{3}{4}t(a^2+b^2+c^2)(x^2+y^2+z^2)$$
$$-\tfrac{1}{2}\{(b-c)(c-a)(a-b)(y-z)(z-x)(x-y)\}$$
$$-\tfrac{27}{2}abcxyz=0,$$

which, on substitution and clearing of fractions, is the relation required.

ON EULER'S IDENTITIES.

By *Prof. M. J. M. Hill*, University College, London.

1. Euler's identities are

$$(1-x)^{\alpha+\beta-\gamma}F(\alpha,\beta,\gamma,x)=F(\gamma-\alpha,\gamma-\beta,\gamma,x)\ldots(\text{i}),$$

and $$F(\alpha,\beta,\gamma,x)=(1-x)^{-\alpha}F\left(\alpha,\gamma-\beta,\gamma,\frac{x}{x-1}\right)\ldots.(\text{ii}).$$

I have not been able to find any simple *algebraic* proof of these two identities, so that it is possible that the following may be useful.

2. Adopting the abbreviation

$$c_n \text{ for } c(c+1)(c+2)\ldots(c+n-1),$$

the binomial expansion for $(1-x)^{\alpha+\beta-\gamma}$ is

$$1+(\gamma-\alpha-\beta)x+\ldots+\frac{(\gamma-\alpha-\beta)_n}{n!}x^n+\ldots.$$

Hence the coefficient of x^n in

$$(1-x)^{\alpha+\beta-\gamma}F(\alpha,\beta,\gamma,x)$$

is $$\frac{\alpha_n\beta_n}{n!\,\gamma_n}+\frac{\alpha_{n-1}\beta_{n-1}}{(n-1)!\,\gamma_{n-1}}(\gamma-\alpha-\beta)+\frac{\alpha_{n-2}\beta_{n-2}}{(n-2)!\,\gamma_{n-2}}\frac{(\gamma-\alpha-\beta)_2}{2!}+\ldots$$

$$+\frac{\alpha_{n-r}\beta_{n-r}}{(n-r)!\,\gamma_{n-r}}\frac{(\gamma-\alpha-\beta)_r}{r!}+\ldots+\frac{(\gamma-\alpha-\beta)_n}{n!}.$$

$$=\frac{1}{n!\,\gamma_n}\left[\begin{array}{l}\alpha_n\beta_n+{}_nC_1\alpha_{n-1}\beta_{n-1}(\gamma+n-1)(\gamma-\alpha-\beta)\\ +{}_nC_2\alpha_{n-2}\beta_{n-2}(\gamma+n-2)_2(\gamma-\alpha-\beta)_2\\ +\ldots\\ +{}_nC_r\alpha_{n-r}\beta_{n-r}(\gamma+n-r)_r(\gamma-\alpha-\beta)_r\\ +\ldots\\ +\gamma_n(\gamma-\alpha-\beta)_n\end{array}\right]$$

To establish the identity (i) it is therefore necessary to prove that the quantity in square brackets $=(\gamma-\alpha)_n(\gamma-\beta)_n$. When $n=1$ the result to be proved is

$$(\gamma-\alpha)(\gamma-\beta)=\alpha\beta+\gamma(\gamma-\alpha-\beta)\ldots\ldots\ldots(1).$$

This is true for all values of α, β, γ.

Replacing α, β, γ in it by $\alpha+r$, $\beta+r$, $\gamma+r$, we have

$$(\gamma-\alpha)(\gamma-\beta)=(\alpha+r)(\beta+r)+(\gamma+r)(\gamma-\alpha-\beta-r)\ldots(\text{II}),$$

which we shall require later.

Now in (I) increase γ by 1, this gives

$$(\gamma-\alpha+1)(\gamma-\beta+1)=\alpha\beta+(\gamma+1)(\gamma-\alpha-\beta+1)\ldots(\text{III}).$$

Multiply each side of (III) by $(\gamma-\alpha)(\gamma-\beta)$.

The left-hand side gives

$$(\gamma-\alpha)_2(\gamma-\beta)_2.$$

The first term on the right-hand side gives

$$(\alpha\beta)[(\gamma-\alpha)(\gamma-\beta)],$$

or, using (II) with the value 1 for r, we get

$$\alpha\beta[(\alpha+1)(\beta+1)+(\gamma+1)(\gamma-\alpha-\beta-1)]$$
$$=\alpha_2\beta_2+\alpha\beta(\gamma+1)(\gamma-\alpha-\beta-1).$$

The second term on the right-hand side of (III) gives

$$(\gamma+1)(\gamma-\alpha-\beta+1)[(\gamma-\alpha)(\gamma-\beta)]$$
$$=(\gamma+1)(\gamma-\alpha-\beta+1)[\alpha\beta+\gamma(\gamma-\alpha-\beta)]$$
$$=\alpha\beta(\gamma+1)(\gamma-\alpha-\beta+1)+\gamma_2(\gamma-\alpha-\beta)_2.$$

Hence, from (III.) it follows that

$$(\gamma-\alpha)_2(\gamma-\beta)_2$$
$$=\alpha_2\beta_2+2\alpha\beta(\gamma+1)(\gamma-\alpha-\beta)+\gamma_2(\gamma-\alpha-\beta)_2.$$

Thus the required formula is proved for $n=2$ for all values of α, β, γ.

Assume then that it has been proved that for all values of α, β, γ and for all integral values of n up to a specified value that

$$\begin{aligned}(\gamma-\alpha)_n(\gamma-\beta)_n=&\alpha_n\beta_n\\&+{}_nC_1\alpha_{n-1}\beta_{n-1}(\gamma+n-1)(\gamma-\alpha-\beta)\\&+{}_nC_2\alpha_{n-2}\beta_{n-2}(\gamma+n-2)_2(\gamma-\alpha-\beta)_2\\&+\ldots\\&+{}_nC_r\alpha_{n-r}\beta_{n-r}(\gamma+n-r)_r(\gamma-\alpha-\beta)_r\\&+\ldots\\&+\gamma_n(\gamma-\alpha-\beta)_n\ldots\ldots\ldots\ldots\ldots\ldots(\text{IV}).\end{aligned}$$

Replace γ by $\gamma+1$; this gives

$$\begin{aligned}(\gamma-\alpha+1)_n(\gamma-\beta+1)_n &= \alpha_n\beta_n \\ &+ {}_nC_1\alpha_{n-1}\beta_{n-1}(\gamma+n)(\gamma-\alpha-\beta+1) \\ &+ {}_nC_2\alpha_{n-2}\beta_{n-2}(\gamma+n-1)_2(\gamma-\alpha-\beta+1)_2 \\ &+ \dots \\ &+ {}_nC_r\alpha_{n-r}\beta_{n-r}(\gamma+n-r+1)_r(\gamma-\alpha-\beta+1)_r \\ &+ \dots \\ &+ (\gamma+1)_n(\gamma-\alpha-\beta+1)_n \quad \dots\dots\dots\dots (V.)\end{aligned}$$

Multiply both sides by $(\gamma-\alpha)(\gamma-\beta)$.

The left-hand side gives

$$(\gamma-\alpha)_{n+1}(\gamma-\beta)_{n+1}.$$

The $(r+1)^{\text{th}}$ term on the right-hand side gives

$${}_nC_r\alpha_{n-r}\beta_{n-r}(\gamma+n-r+1)_r(\gamma-\alpha-\beta+1)_r(\gamma-\alpha)(\gamma-\beta).$$

Take the form given by (II.) for $(\gamma-\alpha)(\gamma-\beta)$ when r is replaced by $n-r$, then we have to multiply

$${}_nC_r\alpha_{n-r}\beta_{n-r}(\gamma+n-r+1)_r(\gamma-\alpha-\beta+1)_r$$

by $(\alpha+n-r)(\beta+n-r)+(\gamma+n-r)(\gamma-\alpha-\beta-n+r)$.

Thus the $(r+1)^{\text{th}}$ term of the right-hand side of (V.) gives

$${}_nC_r\alpha_{n-r+1}\beta_{n-r-1}(\gamma+n-r+1)_r(\gamma-\alpha-\beta+1)_r$$

$$+ {}_nC_r\alpha_{n-r}\beta_{n-r}(\gamma+n-r)_{r+1}(\gamma-\alpha-\beta+1)_r(\gamma-\alpha-\beta-n+r).$$

Hence the r^{th} term on the right-hand side of (V.) gives

$${}_nC_{r-1}\alpha_{n-r+2}\beta_{n-r+2}(\gamma+n-r+2)_{r-1}(\gamma-\alpha-\beta+1)_{r-1}$$

$$+ {}_nC_{r-1}\alpha_{n-r+1}\beta_{n-r+1}(\gamma+n-r+1)_r(\gamma-\alpha-\beta+1)_{r-1}(\gamma-\alpha-\beta-n+r-1).$$

Consequently the coefficient of $\alpha_{n-r+1}\beta_{n-r+1}$ will be

$$\begin{aligned}&{}_nC_r(\gamma+n-r+1)_r(\gamma-\alpha-\beta+1)_r \\ &+ {}_nC_{r-1}(\gamma+n-r+1)_r(\gamma-\alpha-\beta+1)_{r-1}(\gamma-\alpha-\beta-n+r-1) \\ &= (\gamma+n-r+1)_r(\gamma-\alpha-\beta+1)_{r-1}\end{aligned}$$

multiplied by

$$\begin{aligned}&{}_nC_r(\gamma-\alpha-\beta+r)+{}_nC_{r-1}(\gamma-\alpha-\beta-n+r-1) \\ &= ({}_nC_r+{}_nC_{r-1})(\gamma-\alpha-\beta)+\{r({}_nC_r)-(n-r+1)({}_nC_{r-1})\} \\ &= {}_{n+1}C_r(\gamma-\alpha-\beta).\end{aligned}$$

Thus the coefficient of $\alpha_{n+1-r}\beta_{n+1-r}$ is

$$_{n+1}C_r(\gamma-\alpha-\beta)(\gamma+n-r+1)_r(\gamma-\alpha-\beta+1)_{r-1},$$

i.e. $$_{n+1}C_r(\gamma+n-r+1)_r(\gamma-\alpha-\beta)_r.$$

Hence the $(r+1)^{\text{th}}$ term on the right-hand side of the value of $(\gamma-\alpha)_{n+1}(\gamma-\beta)_{n+1}$ is

$$_{n+1}C_r\alpha_{n+1-r}\beta_{n+1-r}(\gamma+n+1-r)_r(\gamma-\alpha-\beta)_r,$$

which is what the $(r+1)^{\text{th}}$ term of the value of $(\gamma-\alpha)_n(\gamma-\beta)_n$ given in (IV.) becomes when n is changed into $(n+1)$.

And since the equation holds good when $n=1$ and 2, it holds good when $n=3, 4, \ldots$, and thus it holds good for any positive integral value of n.

Thus the first identity is proved.

3. To prove the second identity the right-hand side is

$$(1-x)^{-\alpha}F\left(\alpha,\ \gamma-\beta,\ \gamma,\ \frac{x}{x-1}\right)$$

$$=(1-x)^{-\alpha}+\frac{\alpha(\gamma-\beta)}{1.\gamma}(-x)(1-x)^{-\alpha-1}$$

$$+\frac{\alpha_2(\gamma-\beta)_2}{2!\,\gamma_2}(-x)^2(1-x)^{-\alpha-2}$$

$$+\ldots$$

$$+\frac{\alpha_n(\gamma-\beta)_n}{n!\,\gamma_n}(-x)^n(1-x)^{-\alpha-n}$$

$$+\ldots.$$

Hence the coefficient of x^n is

$$\frac{\alpha_n}{n!}+\frac{\alpha(\gamma-\beta)}{1.\gamma}(-1)\frac{(\alpha+1)_{n-1}}{(n-1)!}$$

$$+\frac{\alpha_2(\gamma-\beta)_2}{2!\,\gamma_2}(-1)^2\frac{(\alpha+2)_{n-2}}{(n-2)!}+\ldots+(-1)^n\frac{\alpha_n(\gamma-\beta)_n}{n!\,\gamma_n}$$

$$=\frac{\alpha_n}{n!\,\gamma_n}\left\{\begin{array}{l}\gamma_n-{}_nC_1(\gamma-\beta)(\gamma+1)_{n-1}\\ \quad+{}_nC_2(\gamma-\beta)_2(\gamma+2)_{n-2}+\ldots+(-1)^n(\gamma-\beta)_n\end{array}\right\}.$$

The $(r+1)^{\text{th}}$ term in the brackets is

$$(-1)^r{}_nC_r(\gamma-\beta)_r(\gamma+r)_{n-r}$$

$$=(-1)^r{}_nC_r(\gamma-\beta)(\gamma-\beta+1)\ldots(\gamma-\beta+r-1)$$
$$\times(\gamma+r)(\gamma+r+1)\ldots(\gamma+n-1)$$

$$={}_nC_r(\beta-\gamma)(\beta-\gamma-1)\ldots(\beta-\gamma-r+1)$$
$$\times(\gamma+n-1)(\gamma+n-2)\ldots(\gamma+r).$$

If now we adopt, as in the statement of Vandermonde's theorem, an abbreviation for

$$a\,(a-1)\ldots(a-r+1),$$

say $\overline{a}_r$, so that Vandermonde's theorem can be written

$$\overline{(a+b)}_n = \sum_{r=0}^{n} {}_nC_r\,\overline{a}_r\overline{b}_{n-r}.$$

Then we have, for the $(r+1)^{\text{th}}$ term in brackets, the expression

$${}_nC_r\,\overline{(\beta-\gamma)}_r\,\overline{(\gamma+n-1)}_{n-r};$$

and therefore, by Vandermonde's theorem, the whole expression in brackets

$$\begin{aligned}
&= \overline{(\beta-\gamma+\gamma+n-1)}_n \\
&= \overline{(\beta+n-1)}_n \\
&= (\beta+n-1)\,(\beta+n-2)\ldots(\beta+1)\,\beta \\
&= \beta_n \text{ in the notation of this paper;}
\end{aligned}$$

and therefore the coefficient of x^n is $\dfrac{\alpha_n\beta_n}{n!\,\gamma}$, which is the coefficient of x^n in $F(\alpha, \beta, \gamma, x)$, and thus the second identity is demonstrated.

October 30th, 1914.

NOTES ON LAPLACE'S INTEGRALS FOR $P_n(x)$ AND $Q_n(x)$.

By *F. Jackson, M.A., B.Sc.*

THE following paper gives a simple method for proving that Laplace's integrals for $P_n(x)$ and $Q_n(x)$, viz.

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi} \{x + \sqrt{(x^2-1)}\cos\phi\}^n d\phi,$$

$$Q_n(x) = \int_0^{\infty} \frac{d\phi}{\{x + \sqrt{(x^2-1)}\cosh\phi\}^{n+1}},$$

satisfy Legendre's equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0.$$

Incidentally two of the recurrence formulæ for each of the above functions are proved.

I also show that the method can be applied to the integral expression for the Bessel function, namely

$$\int_0^\pi \cos(n\phi - x\sin\phi)\,d\phi.$$

The usual method, by substituting the integral directly into the left-hand side of the differential equation, involves some very subtle manipulation of the integrand obtained in order to reduce it to a perfect differential.

Consider $$\int_0^\pi \{x+\sqrt{(x^2-1)}\cos\phi\}^n d\phi.$$

Write $$y_n = \int_0^\pi T^n d\phi,$$

where $$T = x+\sqrt{(x^2-1)}\cos\phi,$$

$$\frac{dy_n}{dx} = \int_0^\pi n T^{n-1}\frac{dT}{dx}\,d\phi$$

$$= n\int_0^\pi T^{n-1}\left(1+\frac{x\cos\phi}{\sqrt{(x^2-1)}}\right)d\phi$$

$$= n\int_0^\pi T^{n-1}\left(1+\frac{x(T-x)}{(x^2-1)}\right)d\phi,$$

therefore $$(x^2-1)\frac{dy_n}{dx} = n\int_0^\pi \{(x^2-1)\,T^{n-1}+xT^n-x^2T^{n-1}\}\,d\phi$$

$$= nx\int_0^\pi T^n d\phi - n\int_0^\pi T^{n-1}d\phi,$$

therefore $$(x^2-1)\frac{dy_n}{dx} = nxy_n - ny_{n-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Again $$\int_0^\pi \frac{d}{d\phi}\{\sin\phi\,T^n\}\,d\phi = 0,$$

therefore $$\int_0^\pi [\cos\phi\,T^n + nT^{n-1}\{-\sqrt{(x^2-1)}\sin\phi\}\sin\phi]\,d\phi = 0,$$

therefore $$\int_0^\pi \{\cos\phi\,T^n + nT^{n-1}\sqrt{(x^2-1)}\,(\cos^2\phi-1)\}\,d\phi = 0.$$

Multiply by $\sqrt{(x^2-1)}$ and substitute $\sqrt{(x^2-1)}\cos\phi = T-x$,

therefore $\int_0^\pi \{T^n(T-x) + nT^{n-1}(T-x)^2 - nT^{n-1}(x^2-1)\}d\phi = 0$,

therefore $\int_0^\pi \{(n+1)T^{n+1} - (2n+1)xT^n + nT^{n-1}\}d\phi = 0$,

therefore $(n+1)y_{n+1} - (2n+1)xy_n + ny_{n-1} = 0$.....(2).

Differentiate (1) after multiplying by -1,

$$\frac{d}{dx}\left((1-x^2)\frac{dy_n}{dx}\right) = n\frac{dy_{n-1}}{dx} - ny_n - nx\frac{dy_n}{dx}.$$

Using (1)

$$= \frac{n}{x^2-1}\{(n-1)xy_{n-1} - (n-1)y_{n-2}\} - ny_n - \frac{nx}{x^2-1}(nxy_n - ny_{n-1}),$$

$$= \frac{n}{x^2-1}\{(n-1)xy_{n-1} - (n-1)y_{n-2} + nxy_{n-1}\} - ny_n - \frac{n^2x^2}{x^2-1}y_n,$$

$$= \frac{n}{x^2-1}\{(2n-1)xy_{n-1} - (n-1)y_{n-2}\} - ny_n - \frac{n^2x^2y_n}{x^2-1}.$$

From (2), by changing n into $n-1$, we get

$$ny_n - (2n-1)xy_{n-1} + (n-1)y_{n-2} = 0.$$

therefore $$\frac{d}{dx}\left((1-x^2)\frac{dy_n}{dx}\right) = \frac{n}{x^2-1}.ny_n - ny_n - \frac{n^2x^2y_n}{x^2-1},$$

$$= -n^2y_n - ny_n,$$

$$= -n(n+1)y_n;$$

therefore $$(1-x^2)\frac{d^2y_n}{dx^2} - 2x\frac{dy_n}{dx} + n(n+1)y_n = 0.$$

Now consider the integral

$$\int_0^\infty \frac{d\phi}{\{x+\sqrt{(x^2-1)}\cosh\phi\}^{n+1}},$$

which has a meaning if the real part of $(n+1)$ is positive.

Call it y_n, and let $U = x + \sqrt{(x^2-1)}\cosh\phi$,

therefore $$y_n = \int_0^\infty \frac{d\phi}{U^{n+1}}.$$

Proceeding as before

$$\frac{dy_n}{dx} = -(n+1)\int_0^\infty \frac{d\phi}{U^{n+2}}\left(1 + \frac{x(U-x)}{x^2-1}\right)$$

$$= -\frac{(n+1)}{x^2-1}\int_0^\infty \left(\frac{x}{U^{n+1}} - \frac{1}{U^{n+2}}\right)d\phi,$$

therefore $$(x^2-1)\frac{dy_n}{dx} = (n+1)y_{n+1} - (n+1)xy_n \ldots\ldots\ldots(3).$$

Again, if the real part of n is positive

$$\int_0^\infty \frac{d}{d\phi}\left(\frac{\sinh\phi}{U^{n+1}}\right)d\phi = 0,$$

therefore $$\int_0^\infty \left(\frac{\cosh\phi}{U^{n+1}} - \frac{(n+1)\sqrt{(x^2-1)}\sinh^2\phi}{U^{n+2}}\right)d\phi = 0,$$

from which we easily find

$$n y_{n-1} - (2n+1)x y_n + (n+1)y_{n+1} = 0.$$

Changing n into $n+1$ we see that, if the real part of $(n+1)$ is positive, then

$$(n+1)y_n - (2n+3)x y_{n+1} + (n+2)y_{n+2} = 0 \ldots\ldots\ldots(4).$$

Differentiate (3), therefore

$$\frac{d}{dx}\left((1-x^2)\frac{dy_n}{dx}\right) = (n+1)\left[y_n + x\frac{dy_n}{dx} - \frac{dy_{n+1}}{dx}\right].$$

Use (3)
$$= (n+1)\left[y_n + \frac{x}{x^2-1}(n+1)(y_{n+1} - xy_n) - \frac{n+2}{x^2-1}(y_{n+2} - xy_{n+1})\right]$$

$$= (n+1)\left[y_n - \frac{x^2(n+1)}{x^2-1}y_n + \frac{(2n+3)xy_{n+1} - (n+2)y_{n+2}}{x^2-1}\right].$$

Using (4)
$$= (n+1)\left[y_n - \frac{x^2(n+1)}{x^2-1}\, y_n + \frac{(n+1)\, y_n}{x^2-1}\right]$$
$$= (n+1)\left[y_n - (n+1)\, y_n\right]$$
$$= (n+1)\,(-n y_n)$$
$$= -\,n\,(n+1)\, y_n,$$

so that y_n satisfies Legendre's equation if the real part of $(n+1)$ is positive.

It will be noticed that the following recurrence formulæ have been proved [assuming the integrals have been identified with $P_n(x)$, $Q_n(x)$ respectively]:

$$(x^2-1)\frac{dP_n}{dx} = nxP_n - nP_{n-1} \text{ ....(1)},$$

$$(n+1)\,P_{n+1} - (2n+1)\,xP_n + nP_{n-1} = 0 \text{.....(2)},$$

$$(x^2-1)\frac{dQ_n}{dx} = (n+1)\,Q_{n+1} - (n+1)\,xQ_n \text{ .....(3)},$$

$$(n+1)\,Q_{n+1} - (2n+1)\,xQ_n + nQ_{n-1} = 0 \text{.....(4)}.$$

Now consider $\displaystyle\int_0^\pi \cos(n\phi - x\sin\phi)\, d\phi,$

which equals $\pi J_n(x)$ if n is integral.

Let $\displaystyle y_n = \int_0^\pi \cos(n\phi - x\sin\phi)\, d\phi,$

therefore $\displaystyle\frac{dy_n}{dx} = \int_0^\pi \sin(n\phi - x\sin\phi)\sin\phi\, d\phi,$

$$= \tfrac{1}{2}\int_0^\pi \left[\cos\{(n-1)\phi - x\sin\phi\} - \cos\{(n+1)\phi - x\sin\phi\}\right] d\phi,$$

therefore $\displaystyle\frac{dy_n}{dx} = \tfrac{1}{2}(y_{n-1} - y_{n+1}) \text{ .....(5)}.$

Again $\displaystyle\int_0^\pi \frac{d}{d\phi}\left[\sin(n\phi - x\sin\phi)\right] d\phi = \sin n\pi,$

therefore $\displaystyle\int_0^\pi \cos(n\phi - x\sin\phi)\,(n - x\cos\phi)\, d\phi = \sin n\pi,$

therefore $\displaystyle ny_n - x\int_0^\pi \cos(n\phi - x\sin\phi)\cos\phi\, d\phi = \sin n\pi,$

therefore $\displaystyle ny_n - \frac{x}{2}(y_{n+1} + y_{n-1}) = \sin n\pi \text{.....(6)}.$

From (5)

$$\frac{d}{dx}\left(x\frac{dy_n}{dx}\right) = \tfrac{1}{2}(y_{n-1} - y_{n+1}) + \frac{x}{2}\left(\frac{dy_{n-1}}{dx} - \frac{dy_{n+1}}{dx}\right)$$

$$= \tfrac{1}{2}(y_{n-1} - y_{n+1}) + \frac{x}{4}(y_{n-2} - 2y_n + y_{n+2})$$

$$= \tfrac{1}{2}(y_{n-1} - y_{n+1}) - \frac{xy_n}{2} + \tfrac{1}{2}\left(\frac{x}{2}y_{n-2} + \frac{x}{2}y_{n+2}\right).$$

Now in (6) change n into $(n+1)$ and $(n-1)$,

therefore $\quad (n+1)y_{n+1} - \dfrac{x}{2}(y_{n+2} + y_n) = \sin(n+1)\pi = -\sin n\pi,$

$$(n-1)y_{n-1} - \frac{x}{2}(y_n + y_{n-2}) = \sin(n-1)\pi = -\sin n\pi,$$

Adding

$$(n+1)y_{n+1} + (n-1)y_{n-1} - \frac{x}{2}(y_{n+2} + y_{n-2}) - xy_n = -2\sin n\pi.$$

therefore

$$\frac{d}{dx}\left(x\frac{dy_n}{dx}\right) = \tfrac{1}{2}(y_{n-1} - y_{n+1}) - \frac{xy_n}{2} + \tfrac{1}{2}\{(n+1)y_{n+1} + (n-1)y_{n-1}\}$$

$$- \frac{xy_n}{2} + \sin n\pi$$

$$= \frac{n}{2}(y_{n+1} + y_{n-1}) - xy_n + \sin n\pi$$

$$= \frac{n}{x}(ny_n - \sin n\pi) - xy_n + \sin n\pi,$$

therefore

$$x\frac{d^2y_n}{dx^2} + \frac{dy_n}{dx} = y_n\left(\frac{n^2}{x} - x\right) + \sin n\pi\left(1 - \frac{n^2}{x}\right),$$

therefore

$$\frac{d^2y_n}{dx^2} + \frac{1}{x}\frac{dy_n}{dx} + \left(1 - \frac{n^2}{x^2}\right)y_n = \sin n\pi\left(\frac{1}{x} - \frac{n^2}{x^2}\right),$$

and if n is integral this reduces to Bessel's equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0.$$

A SIMPLE TRANSFORMATION OF CERTAIN PARTIAL DIFFERENTIAL EQUATIONS.

By *J. R. Wilton, M.A., D.Sc.*

A VERY simple transformation is sometimes effective for the solution of partial differential equations which can be expressed in the form

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},$$

where P and Q are functions of x, y, z, p, and q. (We restrict ourselves to the case of equations of the second order.)

We put $P = p'$, $Q = q'$,

where p' and q' are the first order derivatives of a new dependent variable z', and from these two equations we eliminate z and its derivatives. The result will, in general, be that we thus obtain two equations of the third order for z', but in certain cases we may obtain a soluble equation of the second order.

As an example, consider the equation

$$\frac{\partial^2 z}{\partial y^2} + \frac{\partial^2}{\partial x^2}(\log z) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

which is equation (4) of the paper* on "Some simple transformations of Stokes' current function equation."

The most obvious transformation of the kind indicated would be to put

$$\frac{\partial z}{\partial y} = p', \quad \frac{\partial}{\partial x}(\log z) = q';$$

but this leads to two not very simple equations of the third order in z'. We begin by interchanging x and z; equation (1) then becomes

$$\frac{\partial^2 x}{\partial z^2}\left\{\left(\frac{\partial x}{\partial y}\right)^2 + \frac{1}{z}\right\} - 2\frac{\partial x}{\partial y}\frac{\partial x}{\partial z}\frac{\partial^2 x}{\partial y \partial z} + \left(\frac{\partial x}{\partial z}\right)^2 \frac{\partial^2 x}{\partial y^2} + \frac{1}{z^2}\frac{\partial x}{\partial z} = 0,$$

say, $$r(q^2 + 1/x) - 2pqs + p^2 t + p/x^2 = 0.$$

* *Messenger*, vol. xliii., pp. 110–114.

Now

$$\frac{\partial q}{\partial y} - \frac{\partial}{\partial x}\{(q^2 + 1/x)/p\} = \frac{1}{p^2}\{p^2 t + r/x + p/x^2 + q^2 r - 2pqs\} = 0.$$

Hence we may put

$$q = \frac{\partial u}{\partial x}, \quad \left(q^2 + \frac{1}{x}\right) \Big/ p = \frac{\partial u}{\partial y},$$

and the equation for u becomes

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial y}\left\{\left[\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{x}\right] \Big/ \frac{\partial u}{\partial y}\right\}.$$

i.e. it is of the form

$$q^2 r - 2pq s + (p^2 + 1/x)\, t = 0\ ;$$

and, finally, interchanging y and u, it becomes

$$r + t/x = 0.$$

The solution of this equation may be expressed in various forms, of which perhaps the most concise is

$$\iota z = \int_0^\pi x^{\frac{1}{2}} f(2x^{\frac{1}{2}} \cos\lambda + \iota y) \cos\lambda\, d\lambda,$$

where f is an arbitrary functional form. The solution of the original equation (1) is to be obtained from

$$\iota y = z^{\frac{1}{2}} \int_0^\pi f'(2z^{\frac{1}{2}} \cos\lambda + \iota u) \cos\lambda\, d\lambda,$$

$$\frac{\partial x}{\partial y} = \frac{\partial u}{\partial z}, \quad \left[\left(\frac{\partial x}{\partial y}\right)^2 + \frac{1}{z}\right] \Big/ \frac{\partial x}{\partial z} = \frac{\partial u}{\partial y}.$$

The derivatives $\dfrac{\partial u}{\partial y}$ and $\dfrac{\partial u}{\partial z}$ are given by

$$z^{\frac{1}{2}} \frac{\partial u}{\partial y} \int_0^\pi f' \cos\lambda\, d\lambda = 1,$$

$$\iota z^{\frac{1}{2}} \frac{\partial u}{\partial z} \int_0^\pi f' \cos\lambda\, d\lambda = -\int_0^\pi f' \cos^2\lambda\, d\lambda$$

$$-\tfrac{1}{2} z^{-\frac{1}{2}} \int_0^\pi f \cos\lambda\, d\lambda = -\int_0^\pi f'\, d\lambda,$$

so that

$$\frac{\partial u}{\partial z} = \iota \frac{\partial u}{\partial y} \int_0^\pi f'\, d\lambda.$$

Let us try the effect of putting

$$x = \int_0^\pi \phi\,(2z^{\frac{1}{2}}\cos\lambda + \iota u)\,d\lambda,$$

where ϕ is a function whose form depends on that of f. Then

$$\iota\,\frac{\partial u}{\partial y}\int_0^\pi \phi'\,d\lambda = \frac{\partial x}{\partial y} = -\frac{\partial u}{\partial z}.$$

So that this equation is satisfied if $\phi = f$.

Again, putting $\phi = f$, we find

$$\frac{\partial x}{\partial z} = z^{-\frac{1}{2}}\int_0^\pi f'\cos\lambda\,d\lambda + \iota\,\frac{\partial u}{\partial z}\int_0^\pi f'\,d\lambda,$$

i.e.

$$\frac{\partial u}{\partial y}\,\frac{\partial x}{\partial z} = \frac{1}{z} + \left(\frac{\partial u}{\partial z}\right)^2 = \frac{1}{z} + \left(\frac{\partial x}{\partial y}\right)^2,$$

so that the second relation between x and u is also satisfied.

Thus one form of solution of (1) is the result of eliminating u from the equations

$$x = \int_0^\pi f\,(2z^{\frac{1}{2}}\cos\lambda + \iota u)\,d\lambda,$$

$$\iota y = z^{\frac{1}{2}}\int_0^\pi f\,(2z^{\frac{1}{2}}\cos\lambda + \iota u)\cos\lambda\,d\lambda.$$

This form of solution is obtained most readily by reversing the argument by which the equation was deduced in the paper cited above.

Following somewhat the same line of argument as that given above, another solution of equation (1) may be made to depend on that of the equation $t + \log r = 0$. For on account of the form of (1) we may evidently put

$$-z = r', \quad \log z = t',$$

so that

$$r' = e^{-t'}.$$

The solution of this equation may be written down*. It is the result of eliminating X and Y from

$$x = -\int_a^b F_1(\mu)\,I_0(2\mu e^{-\frac{1}{2}X})\sin\mu Y d\mu + \int_a^b F_2(\mu)\,G_0(2\mu e^{-\frac{1}{2}X})\cos\mu Y d\mu \ldots (2),$$

* Note on the solution of the differential equation $r = f(t)$. *Messenger*, vol. xliii, p. 62.

$$y = -e^{-\frac{1}{2}X}\left\{\int_a^b F_1(\mu)\, I'_0(2\mu e^{-\frac{1}{2}X})\cos\mu Y d\mu + \int_a^b F_2(\mu) G'_0(2\mu e^{-\frac{1}{2}X})\sin\mu Y d\mu\right\} \ldots (3),$$

and an expression for z which it is unnecessary to write down, where F_1 and F_2 are arbitrary functions of μ, and I_0 and G_0 are Bessel's functions as defined in *Gray & Mathews' Bessel Functions*, pp. 66 and 91.

To obtain the solution of (1) we have to eliminate X and Y from (2), (3), and

$$z = -r' = -e^{-X}.$$

By a slight change of the variable μ we may express the solution of (1) as the eliminant of θ from

$$x = \int_{-\infty}^{\infty} F(\lambda)\, J_0(\lambda z^{\frac{1}{2}})\sinh\lambda\theta\, d\lambda,$$

$$y = z^{\frac{1}{2}}\int_{-\infty}^{\infty} F(\lambda) J'_0(\lambda z^{\frac{1}{2}})\cosh\lambda\theta\, d\lambda,$$

where $F(\lambda)$ is restricted only by the condition that the integrals must converge. It is unnecessary to include the integrals containing K_0 and K'_0.

The solution of the more general equation,

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2}{\partial y^2}\{f(z)\},$$

may evidently be written down in the same way. In fact, if ϕ_1 and ϕ_2 are two linearly independent solutions of the equation

$$\phi''(z) + \mu^2 f(z)\,\phi(z) = 0,$$

the general solution of

$$r - tf(z) - q^2 f'(z) = 0$$

is the result of eliminating θ between the two equations

$$x = -\int_a^b \mu F_1(\mu)\,\phi_1 \sin\mu\theta\, d\mu + \int_a^b \mu F_2(\mu)\,\phi_2 \cos\mu\theta\, d\mu,$$

$$y = \int_a^b F_1(\mu)\,\phi_1' \cos\mu\theta\, d\mu + \int_a^b F_2(\mu)\,\phi_2' \sin\mu\theta\, d\mu,$$

where F_1 and F_2 are arbitrary functions of μ.

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A NOTE ON POTENTIAL FUNCTIONS IN FREE SPACE OF n DIMENSIONS.

By *Horace Todd.*

We purpose to find the general solution of

$$\nabla^2 V \equiv \frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial x_2^2} + \ldots + \frac{\partial^2 V}{\partial x_n^2} = 0,$$

and to find whether or no a geometrical interpretation of this solution is possible.

Let $f(x_1, x_2, \ldots, x_n, \lambda) = 0$ be an equipotential locus. Then V is a function of λ only $\equiv \phi(\lambda)$. That is

$$\frac{\nabla^2 \lambda}{\Sigma(\partial\lambda/\partial x_r)^2} = -\frac{\partial}{\partial\lambda} \log \frac{\partial V}{\partial \lambda}.$$

So the fact that if $\Sigma(\partial\lambda/\partial x_r)^2 = 0$ then $\nabla^2\lambda = 0$, leads us to examine solutions of the first equation.

Now the general linear solution of $\Sigma(\partial\lambda/\partial x_r)^2 = 0$ is

$$\lambda = \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n \ (+ \text{ const.}),$$

where $\lambda_1^2 + \lambda_2^2 + \ldots + \lambda_n^2 = 0.$

So we can write

$$\begin{aligned}
\lambda_1 &= i,\\
\lambda_2 &= \sin t_1,\\
\lambda_3 &= \cos t_1 \sin t_2,\\
\lambda_4 &= \cos t_1 \cos t_2 \sin t_3,\\
&\ldots\ldots\ldots\ldots\ldots\ldots,\\
\lambda_{n-1} &= \cos t_1 \cos t_2 \ldots \cos t_{n-3} \sin t_{n-2},\\
\lambda_n &= \cos t_1 \cos t_2 \ldots \cos t_{n-3} \cos t_{n-2}.
\end{aligned}$$

That is, we have expressed the λ's in terms of $(n-2)$ arbitrary parameters. But $V = \phi(\lambda)$ is a solution of $\nabla^2 V = 0$, therefore

$$V = \phi(\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n, \ t_1, \ t_2, \ \ldots, \ t_{n-2})$$

is a solution.

However, it is not sufficiently arbitrary for a general solution, but we can add one degree of functional arbitrariness by integrating ϕ with respect to t_r between convenient constant limits. Doing so in turn for each parameter we see that finally we have a function of $(x_1, \ldots, x_n)$ arbitrary to $(n-1)$ degrees: hence it is the most general solution to $\nabla^2 V = 0$. So

$$V = \int_0^{2\pi}\int_0^{2\pi} \cdots \int_0^{2\pi} \phi(\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n, t_1, t_2, \ldots, t_{n-2})\, dt_1 dt_2 \ldots dt_{n-2},$$

is the required solution.

We proceed to frame an interpretation of this result when $n=3$, *i.e.*, for the case of ordinary space. In this case

$$V=\int_0^{i\pi} f(x\cos t+y\sin t+iz,\ t)\,dt.$$

Consider an element of this sum, say

$$v=f(x\cos t+y\sin t+iz,\ t).$$

Then v is a solution of $\nabla^2 V=0$ and, moreover,

$$\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2=0;$$

moreover, $v=$ const. over the plane $x\cos t+y\sin t+iz=$ const.

So we may say that the most general field is the sum of an infinite number of elementary fields which each have the property that all their equipotential surfaces are tangent planes to Γ, the imaginary circle at infinity, and further that the resultant intensity at every non-singular point of each elementary field is zero.

The generalization to n dimensions is too obvious to need labouring.

This solution points the way to the general solution of $f\left(\frac{\partial}{\partial x_1},\ \ldots,\ \frac{\partial}{\partial x_n}\right)V=0$, where f is a homogeneous function of the symbols $\left(\frac{\partial}{\partial x_1},\ \ldots,\ \frac{\partial}{\partial x_n}\right)$. In fact, we choose n quantities $\lambda_1, \lambda_2, \ldots, \lambda_n$, subject to the equation $f(\lambda_1, \ldots, \lambda_n)=0$, then these can be expressed in terms of $(n-2)$ parameters $t_1, \ldots, t_{n-2}$, and the final solution is

$$V=\int_{T_1}^{T'_1}\int_{T_2}^{T'_2}\cdots\int_{T_{n-2}}^{T'_{n-2}}$$

$$\times\,\phi\,(\lambda_1x_1+\lambda_2x_2+\ldots+\lambda_nx_n,\ t_1,\ t_2,\ \ldots,\ t_{n-2})\,dt_1dt_2\ldots dt_{n-2},$$

where $T_1, T_2, \ldots, T_{n-2}, T'_1, \ldots, T'_{n-2}$ are constants subject to the inequality $T'_r\neq T''_r$ $(r=1, 2, \ldots, n-2)$.

The work is practically a repetition of that for the case of $\nabla^2 V=0$, consisting of finding the linear solution of

$$f\left(\frac{\partial V}{\partial x_1},\ \frac{\partial V}{\partial x_2},\ \ldots,\ \frac{\partial V}{\partial x_n}\right)=0,$$

and adapting it to

$$f\left(\frac{\partial}{\partial x_1},\ \frac{\partial}{\partial x_2},\ \ldots,\ \frac{\partial}{\partial x_n}\right)V=0.$$

THE SEVEN-FOURTEEN SECTION OF THE ZETA FUNCTION.

By *J. R. Wilton, M.A., D.Sc.*

1. The following investigation was undertaken at the request of Sir George Greenhill, and throughout the work I have had the great advantage of his advice and help. The algebra involved requires constant manipulation in order to prevent it from becoming extremely heavy, and Sir George Greenhill's suggestions have frequently prevented me from spending much unnecessary time and labour on the reduction.

The fourteen-section of the zeta function is required to complete the investigation sketched in § 43, p. 290, of Sir George Greenhill's paper "On the Third Elliptic Integral and the Ellipsotomic Problem," *Phil Trans*, vol. cciii. (1904), pp. 217–304, which will be referred to as *Phil. Trans.*

In the paragraph referred to it is required to determine the form of the pseudo-elliptic integral, in the circular form, when the parameter v is equal to $K+fK'i$, where

$$f=\frac{2r+1}{14} \quad (r=1, 2, \ldots, 6).$$

The integral is needed in a dynamical application,* and it can be shown, as on p. 289 of *Phil. Trans.* for the ten-section, to be equal to

$$I(v)=\frac{1}{7}\cos^{-1}\frac{R_0+R_1x+R_2x^2+R_3x^3+R_4x^4+R_5x^5+x^6}{(D-x^2)^{\frac{7}{2}}}\sqrt{(\tfrac{1}{2}X_1)}-\frac{\frac{1}{4}\pi}{7}$$

$$=\frac{1}{7}\sin^{-1}\frac{R_0-R_1x+R_2x^2-R_3x^3+R_4x^4-R_5x^5+x^6}{(D-x^2)^{\frac{7}{2}}}\sqrt{(\tfrac{1}{2}X_2)}-\frac{\frac{1}{4}\pi}{7},$$

$$I(v)=\int_0^x\left(\frac{Q}{D-x^2}-P\right)\frac{dx}{\sqrt{(X_1X_2)}},$$

where

$$X_1,\ X_2=\frac{1}{C^2}\mp 2x-x^2,$$

$$C=\frac{\operatorname{zn}\frac{1}{2}K'}{\sqrt{\kappa}}=\tfrac{1}{2}\left(\frac{1}{\sqrt{\kappa}}-\sqrt{\kappa}\right),$$

* The details of the dynamical application will be found in Sir George Greenhill's *Report on Gyroscopic Theory*, published by H.M. Stationery Office since the present paper was sent to press. The work here undertaken is suggested in § 27, p. 144. The complete analysis required for the three-, four-, five-, and six-sections is given in §§ 25–30, pp. 142–147.

$\sqrt{\kappa}$ being the octahedron irrationality (p. 284),

$$D=\frac{\mathrm{dn}^2 fK'}{\kappa} \text{ or } -\kappa\,\mathrm{tn}^2 fK',$$

$$P=\frac{\mathrm{zn}\, fK'}{\mathrm{zn}\,\frac{1}{2}K'} \text{ or } \frac{\mathrm{zs}\, fK'}{\mathrm{zn}\,\frac{1}{2}K'},$$

$$\frac{Q}{D}=\frac{\mathrm{zn}\, fK'+\mathrm{zn}\,(1-f)\,K'}{\mathrm{zn}\,\frac{1}{2}K'}=\frac{\kappa'^2\,\mathrm{sn}\, fK'\,\mathrm{sn}\,(1-f)\,K'}{\frac{1}{2}(1-\kappa)},$$

or

$$\frac{\mathrm{zs}\, fK'+\mathrm{zs}\,(1-f)\,K'}{\mathrm{zn}\,\frac{1}{2}K'}=\frac{1}{\frac{1}{2}(1-\kappa)\,\mathrm{sn}\, fK'\,\mathrm{sn}\,(1-f)\,K'}.$$

We have $P=P(v)/P(7v)$, where $P(v)$ is practically a zeta function [*Phil. Trans.*, p. 219, equation (7)]; and, when $P(v)$ has been found, the final form of the integral $I(v)$ may be determined by straightforward differentiation and comparison of coefficients. We therefore proceed to the calculation of this function in terms of the quantities* a, b, and e, defined *Phil. Trans.*, p. 285, equations (18) and (19); p. 290, equation (8).

The values of D and Q/D in terms of these quantities are†

$$D=\left(\frac{a-b}{a+b}\right)^2, \quad \frac{Q}{D}=\frac{a^2+b^2}{a^2e},$$

and the value of the leading coefficient in the integral $I(v)$ may be shown to be

$$R_5=7P-1=N\{7P(v)-P(7v)\}$$

in the notation of the next section.

2. To determine $P(v)$ we make use of equation‡ (13), p. 221, of *Phil. Trans.*, namely,

$$N\{7P(v)-P(7v)\}=N\left\{2+y+\frac{y\,(y-x)^2}{(y-x)\,x-y^3}\right\}\ldots(1),$$

where $N=1/P(7v)$, and x and y are defined by Halphen, *Fonctions Elliptiques*, p. 103 (*Phil. Trans.* p. 219, § 2).

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* There is a slight departure from the notation of *Phil. Trans.* We use e instead of c in the equations of p. 290.

† *Phil. Trans.*, p. 285, equation (18), and $D=c^2$; also equation (16), in which $\delta=D/P(7v)$, and $Q(v)=-\frac{1}{2}x$, where x and the a and m which occur in (16) are defined immediately below.

‡ The values of $\gamma_1, \gamma_2, \ldots, \gamma_{11}$ are given in *Proc. L.M.S.*, t. xxv., p. 201. There is, however, a misprint of y^2 instead of y^3 in γ_7, and of $(y-x)^2$ instead of $(y-x)^3$ in γ_{11}.

The connection with a, b, and e is given by the equations

$$x = -\frac{m^3\alpha}{(\alpha - m)^3}, \quad y = -\frac{(1-2m)\,\alpha}{\alpha - m} \ldots\ldots\ldots(2)$$

[*Phil. Trans.*, p. 247, equation (4)],

$$1 - 2m = \frac{a - b^2}{b\,(1-a)}, \quad m = -\frac{(a-b)\,(1+b)}{2b\,(1-a)} \ldots\ldots(3),$$

$$\alpha - m = \frac{(a-b)^4}{8ab\,(a^2+b^2)}, \quad \alpha = -\frac{(a^3-b^2)\,B}{8ab\,(1-a)\,(a^2+b^2)} \ldots(4),$$

where $$B = 3a^2 + a^3 + (1+3a)\,b^2 \ldots\ldots\ldots\ldots(5)$$

[*Phil. Trans.*, p. 285, equations (19), (21), and (22)],

$$P(7v) = C_1 \quad [\text{p. 284, (6)}],$$

$$C_1^{\,2} = \frac{4a^3(a^3-b^4)\,(1+b)^2}{(1-a)^3\,(a-b)^6} \ldots\ldots\ldots\ldots(6)$$

[p. 286, (31)],* and for $\mu = 28$,—*i.e.*, the fourteen-section,—(§ 43, p. 290)

$$ab^5 + (1 + a - a^2 - 3a^3)\,b^4 - a^3(1 + a - 2a^2 - a^3) = 0 \ldots(7).$$

The auxiliary variable e is introduced by means of the equation

$$a^3 - b^4 = a^3(1-a)\,e^2 \ldots\ldots\ldots\ldots(8)$$

[p. 290, (7) and (8)], whence

$$a^2e^2 + ae + a + e = 0 \ldots\ldots\ldots\ldots(9)$$

[p. 291, (10)].

We consider separately the three terms

$$2N, \; Ny, \text{ and } K \equiv \frac{Ny\,(y-x)^2}{(y-x)\,x - y^3}.$$

By means of (6) and (8), we find

$$N = \frac{(1-a)(a-b)^3}{2a^3(1+b)\,e} \ldots\ldots\ldots\ldots(10),$$

and, by eliminating e^2 from (8) and (9),

$$b^4 = a^2 + ae\,(1-a^2) \ldots\ldots\ldots\ldots(11),$$

a relation which will constantly be used.

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* There is a misprint of 16 instead of 4 in this equation.

Taking first the term $2N$, we have

$$a^3e.2N=\frac{1-a}{1+b}\{1+a-(1+b)\}^3$$

$$=\frac{(1-a^2)(1+a)^2}{1+b}-1-2a+3a^3+(1+2a-3a^2)b-(1-a)b^2.$$

But

$$\frac{(1-a^2)(1+a)^2}{1+b}=\frac{(1-a^2)(1+a)^2}{(1-a^2)(1-ae)}(1-b+b^2-b^3)\ \text{[from (11)]}$$

$$=\frac{(1+a)^2(1+ae)}{(1+a)(1+e)}(1-b+b^2-b^3)\ \text{[from (9)]};$$

and $(1+a)(1+ae)=(1+a)(1+ae)+a^2e^2+ae+a+e$

$$=(1+e)(1+2a+a^2e).$$

Therefore, finally,

$$2N=-\frac{2+a}{eb}-\frac{1}{ab}+\frac{1}{a}+\frac{3}{e}-\frac{3+e}{ae}b+\left(\frac{3}{a^2e}+\frac{1}{a}\right)b^2\ldots(12).$$

Taking now the term Ny, we find

$$Ny=\frac{(a+b)(a-b^2)B}{2a^3be(1-a)(1+b)},$$

and, from (11),

$$(a-b^2)B=a(1-a^2)\{(ae-1)^2-1+b^2\},$$

so that

$$2a^2be.Ny=\frac{(a+b)(1+a)}{1+b}(1-ae)^2-(1+a)(1-b)(a+b);$$

and the first of these two terms is equal to

$$\frac{1+a}{1+b}(1-ae)^2\{1+b-(1-a)\}.$$

Of these the term

$$-\frac{1-a^2}{1+b}(1-ae)^2=-(1-ae)(1-b+b^2-b^3),$$

and we have, after a slight reduction by means of (9),

$$Ny=\left(-\frac{1-e}{2}+ae\right)\frac{1}{b}+\frac{1}{2e}-\frac{1}{2a}+\frac{1+e}{2ae}b+\frac{1-ae}{2a^2e}b^2\ldots(13).$$

We proceed now to the heaviest part of the work, the calculation of K. We readily find

$$K=\frac{Ny\,(y-x)^2}{(y-x)\,x-y^3}=-\frac{(a-b^2)\,B(a-b)\,G^2}{2a^3be\,(1-a)\,(1+b)\,F},$$

where

$$G=b\,(a-b)\,(a-b^2)\,(1-a)^2+a\,(1+b)^3\,(a^2+b^2),$$

$$F=a\,(a-b)\,(a^2+b^2)\,(1+b)^3\,G+b\,(1-a)^2\,(a+b)\,(a-b^2)^3\,B.$$

By a straightforward reduction, we find

$$G=(a+b)\,H=(a+b)\,\{a^2+(1+a)\,(a^2+ab+b^2)\,b+ab^4\},$$

$$F=(a+b)^2\{a^4-[a^2(1-2a-a^2)+(1+2a-a^2)\,b^4]\times b\,(a^3+ab+b^2)-a^2b^8\},$$

and, making use of (7),

$$F=-(a+b)^2(a^2+b^2)\,E,$$

where

$$E=a^2(1-2a-a^2)+(1+2a-a^2)\,b^4.$$

Thus

$$K=\frac{(a-b^2)\,B.[(a-b)\,H]^2}{2a^3be\,(1-a)\,(1+b)\,(a^4-b^4)\,E}.$$

By means of (9) and (11), we easily find

$$(a-b^2)\,B=a\,(1-a^2)\,\{(1-ae)^2-(1-b^2)\},$$

$$(a-b)\,H=a^2\,(1-a^2)\,(1+e)\,(ae-b),$$

$$(a^4-b^4)\,E=-a^3e^2\,(1-a^2)^3\,(1-ae),$$

$$1-ae=(1+a)\,(1+e)\,/\,(1+ae).$$

Hence, after a slight reduction by means of (9),

$$K=\frac{(ae-b)^2\,\{(1-ae)^2-(1-b^2)\}}{2a^2be^2\,(1+b)},$$

so that

$$2a^2be^2K=(1-ae)^2\,\frac{\{1+ae-(1+b)\}^2}{1+b}-(ae-b)^2(1-b)$$

$$=\frac{1-a^2e^2}{1+b}\,(1+e)\,(1+a)-2\,(1-ae)^2\,(1+ae)+(1-ae)^2(1+b)-(ae-b)^2(1-b).$$

The first term of this expression is equal to

$$\frac{(1+ae)\,(1+e)}{1-a}\,(1-b+b^2-b^3)=(1+ae^2)\,(1-b+b^2-b^3).$$

And, after a slight further reduction,

$$K=\left(2+\frac{3}{2a}+\frac{1}{e}\right)\frac{1}{b}+1-\frac{1}{2a}+\frac{e-2}{2ae}\,b-\frac{1}{2a}b^2\ldots(14).$$

Finally, adding (12), (13), and (14), we obtain

$$N\{7P(v)-P(7v)\}$$

$$=-\left(\frac{1}{a}+\frac{2+a}{e}\right)\frac{1}{b}+1+\frac{7}{2a^2e}(a^2-ab+b^2)-\frac{b^3}{2a^3e},$$

where use has been made of (11).

The final expression in terms of the single quantity a is somewhat complicated. It can be written down from the relations

$$e=\frac{-1-a+\sqrt{A}}{2a^2},$$

$$b^4=\tfrac{1}{2}a\{-1-a+a^2+3a^3+(1-a^2)\sqrt{A}\},$$

where $A=(1+a)^2-4a^3$

[*Phil. Trans.*, p. 291, equations (11) and (12), and equation (11) of this paper].

A quadric transformation,* in which

$$\kappa=\frac{1-\gamma'}{1+\gamma'},\quad C=\frac{\gamma'}{\gamma},\quad G'=\tfrac{1}{2}(1+\kappa)K',$$

gives $$\frac{Q}{D}=\frac{\mathrm{zn}\,fK'+\mathrm{zn}\,(1-f)K'}{\mathrm{zn}\,\frac{1}{2}K'}=2\,\mathrm{sn}\,2fG',$$

or $$\frac{\mathrm{zs}\,fK'+\mathrm{zs}\,(1-f)K'}{\mathrm{zn}\,\frac{1}{2}K'}=\frac{2}{\gamma'\,\mathrm{sn}\,2fG'},$$

and $$\frac{\mathrm{zn}\,fK'-\mathrm{zn}\,(1-f)K'}{\mathrm{zn}\,\frac{1}{2}K'}=\frac{2\,\mathrm{zn}\,2fG'}{\gamma'},$$

or $$\frac{\mathrm{zs}\,fK'-\mathrm{zs}\,(1-f)K'}{\mathrm{zn}\,\frac{1}{2}K'}=\frac{2\,\mathrm{zs}\,2fG'}{\gamma'},$$

$$\frac{7\,\mathrm{zs}\,2fG'}{\gamma'}=\left(\frac{1}{a}+\frac{2+a}{e}\right)b+\frac{7b}{2ae}+\frac{b^3}{2a^3e};$$

so that a comparison can be made with the results for the *seven-section*, given in § 31 of *Phil. Trans.*

* This note and the value of $I(v)$ in § 1 are supplied by Sir George Greenhill.

The sixteen-section.

3. The corresponding reduction for the case when

$$v = K + \frac{2r+1}{16} K'i,$$

is somewhat simpler. It has been worked independently by two of my pupils at the University of Sheffield, Mr. Hedley Abson and Mr. Alan Cooke, and also by myself.

In this case we require [*Phil. Trans.*, p. 221, (14)],

$$N\{8P(v) - P(8v)\} = N\left\{\tfrac{1}{2}(1+y) + 2 + \frac{(y-x)^2(y-x-y^2)}{y[(y-x)(-y+2x)-xy^2]}\right\},$$

as a function of c, where x and y are given, as before, by equations (2),

$$N = \frac{1}{\sqrt{(C_0)}} = \frac{m-\alpha}{m\alpha},$$

$$m - m^2 = \frac{b(1+b)}{(1+2b)^2 - a^4},$$

$$\frac{1}{\alpha^2} = \frac{(1-a^4)\{(1+2b)^2 - a^4\}}{b^2},$$

[*Phil. Trans.*, p. 295, equations (14), (21) and (24)]; and the relations between a, b and c, are (§ 49)

$$a^4c - (c^4 + 2c^3 - 1)a^2 + c = 0 \quad \ldots\ldots\ldots\ldots (15),$$

$$\left(a + \frac{1}{a}\right)^2 = \frac{c^3+1}{c}(c^2 + 2c - 1), \quad \left(a - \frac{1}{a}\right)^2 = \frac{c-1}{c}(c+1)^3 \ldots (16),$$

$$2a\sqrt{c} = \sqrt{\{(c^2+1)(c^2+2c-1)\}} + (c+1)\sqrt{(c^2-1)} \ldots (17),$$

$$b = ac \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18),$$

[*Phil. Trans.*, p. 301, equations (6), (8), (9), (10) and (11)].

In this case the corresponding expression Q/D is equal to

$$\frac{\sqrt{B}}{2ab} = \frac{c+1}{2c^2}(c^4-1)^{\frac{1}{2}}(c^2-2c-1)^{\frac{1}{2}}$$

$$\times \left\{1 + \frac{2c^{\frac{3}{2}}}{(c^2+1)^{\frac{1}{2}}(c^2+2c-1)^{\frac{1}{2}}}\right\}^{\frac{1}{2}} \left\{1 - \frac{2c^{\frac{3}{2}}}{(c+1)(c^2-1)^{\frac{1}{2}}}\right\}^{\frac{1}{2}} \ldots (19),$$

i.e.,

$$\sqrt{B} = \frac{b}{\alpha}.$$

Hence $$m = \tfrac{1}{2}\left\{1 + \frac{a^4-1}{\sqrt{B}}\right\}, \quad 1-2m = \frac{1-a^2}{\sqrt{B}},$$

$$\frac{2\sqrt{B}}{ab} = \frac{2}{a\alpha} = \frac{2m}{m-\alpha}N.$$

And therefore,

$$A = N\{8P(v) - P(8v)\} - \frac{2\sqrt{B}}{ab}$$

$$= N\left\{\frac{5}{2} + \tfrac{1}{2}y - \frac{2m}{m-\alpha} + \frac{(y-x)^2(y-x-y^2)}{y[(y-x)(-y+2x)-xy^2]}\right\}.$$

In this expression the first three terms are equal to

$$\frac{1}{2a\alpha} - \frac{2}{m\alpha} - \frac{1}{a}.$$

To calculate the remaining term we find without difficulty that, if

$$E \equiv a^4 - (1+2b)^2,$$

$$ma\alpha = \frac{ab}{2B}(\sqrt{B} + a^4 - 1),$$

$$2E\sqrt{B}(m-\alpha)^2(y-x) = b\{(1+b)(a^4-1) + (1-b)\sqrt{B}\},$$

$$2E\sqrt{B}(m-\alpha)^2(y-2x) = (a^4-1-2b^2)\sqrt{B} + (a^4-1)(a^4-1-2b-2b^2),$$

$$2E\sqrt{B}(m-\alpha)^2(y-x-y^2) = b(1-b)(\sqrt{B}+1-a^4),$$

and $$E^3(m-\alpha)^4[(y-x)(-y+2x)-xy^2] = -b^3(a^4+1-2b^2).$$

The first three terms in the expression for A are easily found to be

$$N\left\{\tfrac{1}{2}(1+y) + 2 - \frac{2}{Na\alpha}\right\} = \frac{2(a^4-1) - 10b - 10b^2 - (1-b)\sqrt{B}}{2ab(1+b)}.$$

And, after reduction by means of (15) and (18) we find that $A = P/Q$, where

$$Q = 4ab^3(1+b)(a^4+1-2b^2),$$

$$P = -(1-b)(a^4-1-2b-2b^2-\sqrt{B})$$
$$\times\{(1+b^2)(a^4-1) - 2b(1+b)(1-b)^2 + (1-b^2)\sqrt{B}\}$$
$$+ 2b^3(a^4+1-2b^2)\{2(a^4-1) - 10b - 10b^2 - (1-b)\sqrt{B}\}.$$

The coefficient of $\sqrt{B}$ in P is easily seen to vanish, so that A is, as it should be, rational in a and b. And after further reduction we find

$$P = 2b^2(1+b)$$
$$\times \{(a^4-1)^2 + 2(1-4b-2b^2)(a^4-1) - 2(1-b^2)(1+8b+b^2)\}$$
$$= 2b^2(1+b)\{(a^4-2-8b-b^2)(a^4+1-2b^2) + (1-b^2)(a^4-1)\}.$$

Thus
$$A = P/Q$$
$$= \frac{a^4-2-8b-b^2}{2ab} + \frac{(1-b^2)(a^4-1)}{2ab(a^4+1-2b^2)}$$
$$= \frac{2a^2}{c} - \frac{4}{a} - \frac{3c^4+8c^3+2c^2-3}{2c^4},$$

on making use of (15) and (18). Then, from (17), we find

$$A = \{(c^3+1)^{\frac{1}{2}}(c^2+2c-1)^{\frac{1}{2}} + 2c^{\frac{3}{2}}\}\{(c+1)(c^2-1)^{\frac{1}{2}} - 2c^{\frac{3}{2}}\}/c^2$$
$$-\tfrac{1}{2}c^2 + 2c - 1 + 1/2c^3 \ldots\ldots\ldots\ldots\ldots\ldots (20).$$

Finally, the desired expression for

$$N\{8P(v) - P(8v)\} = A + \frac{\sqrt{B}}{2ab}$$

is given by the sum of the two expressions (19) and (20).

4. There remains for consideration *the eighteen-section*. In this case [*Phil. Trans.*, p. 221, (15)] we require the value of

$$N\{9P(v) - P(9v)\} = N\left\{3 + \frac{(y-x-y^2)^3}{y^3(y-x-y^2) - (y-x)^3}\right\} \ldots (21).$$

As before, x and y are given in terms of m and α by equation (2). And, as in the case of $\mu = 28$,

$$N = 1/C_1 = 1/P(9v),$$

where C_1 is given by equation (6). The relation between a and b is (§ 44, p. 292)

$$a^3b^{12} + (1+3a-6a^3-a^6)b^8 - 3(1+2a-2a^2-2a^3)a^3b^4$$
$$+ (3-3a^2-a^3)a^6 = 0 \ldots (22),$$

and the value of (21) is required in terms of the parameter* r, where

$$a^3-b^4=\frac{a^3(1-a)}{(c+1)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (23),$$

$$c=(1+a)\,r \quad \ldots\ldots\ldots\ldots\ldots\ldots (24),$$

$$r^2a^2-(r^3-r^2-2r-1)\,a-(r+1)(r^2+r+1)=0 \ldots (25),$$

$$a=\frac{r^3-r^2-2r-1+\sqrt{R}}{2r^2} \quad \ldots\ldots\ldots\ldots (26),$$

where $R=r^6+2r^5+5r^4+10r^3+10r^2+4r+1,$

and $$2rc=r^3+r^2-2r-1+\sqrt{R} \quad \ldots\ldots\ldots\ldots (27).$$

The work for this case of the *eighteen-section* (now completed), as also for the *twenty-* and *twenty-two sections*, as well as some numerical verification, is reserved for a subsequent communication.

The University, Sheffield.

THE INVERSION OF THE INTEGRAL

$$u=\int\frac{y\,dx-x\,dy}{\sqrt{(a,b,c,d,e)(x,y)^4}}.$$

By *L. J. Mordell.*

Let $f(x,y)$ be the binary quartic $f(x,y)=ax^4+4bx^3y+\ldots$ whose Hessian is taken to be $h(x,y)=(b^2-ac)\,x^4+\ldots$, and whose sextic covariant is denoted by $g(x,y)$. Suppose p and q are particular values of x and y, and write f for $f(p,q)$, h for $h(x,y)$, etc. Then I have shown† that if $f(x,y)$ is a perfect square for $x=p$, $y=q$ (we are only concerned with the ratio of x to y), the general values of x and y satisfying the same condition are given by

$$\left.\begin{aligned} x&=pt\sqrt{f}-\tfrac{1}{2}s\frac{\partial f}{\partial q}+\tfrac{1}{2}\frac{\partial h}{\partial q}\\ y&=qt\sqrt{f}+\tfrac{1}{2}s\frac{\partial f}{\partial p}-\tfrac{1}{2}\frac{\partial h}{\partial p}\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (1),$$

* A change of notation from q to r is here made in order to avoid confusion with the work of *Phil. Trans.*, § 32, the q in which is connected with r by the relation $q=-\{1/(r+1)\}$.

† *Quarterly Journal of Mathematics*, vol. xlv. (1914), p. 180.

where $t^2 = 4s^3 - g_2 s - g_3$ (all the letters are supposed to denote rational quantities), and then

$$\sqrt{f^3}\sqrt{f}(x, y) = (g + t\sqrt{f^3})^2 - 4(h - sf)^2(h + 2sf).$$

We shall now show that these values of x and y are expressed very simply in terms of u. Hereafter the letters need not denote rational quantities. We have

$$y\,dx - x\,dy = \left(qt\sqrt{f} + \tfrac{1}{2}s\frac{\partial f}{\partial p} - \tfrac{1}{2}\frac{\partial h}{\partial p}\right)\left(p\sqrt{f}\,dt - \tfrac{1}{2}\frac{\partial f}{\partial q}\,ds\right)$$

$$-\left(pt\sqrt{f} - \tfrac{1}{2}s\frac{\partial f}{\partial q} + \tfrac{1}{2}\frac{\partial h}{\partial q}\right)\left(q\sqrt{f}\,dt + \tfrac{1}{2}\frac{\partial f}{\partial p}\,ds\right),$$

and remembering that

$$4f = p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q},$$

$$8g = \frac{\partial f}{\partial p}\frac{\partial h}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial h}{\partial p},$$

this reduces to

$$dt(2s\sqrt{f^3} - 2h\sqrt{f}) - ds(2t\sqrt{f^3} + 2g),$$

and since $2t\,dt = (12s^2 - g_2)\,ds,$

$$t^2 = 4s^3 - g_2 s - g_3,$$

we have $$du = \frac{ds\sqrt{f^3}}{t}$$

$$\times\left(\frac{(12s^2 - g_2)(s\sqrt{f^3} - h\sqrt{f}) - 2(4s^3 - g_2 s - g_3)\sqrt{f^3} - 2gt}{(g + t\sqrt{f^3})^2 - 4(h - sf)^2(h + 2sf)}\right),$$

and, since $$g^2 = 4h^3 - g_2 hf^2 - g_3 f^3,$$

the denominator becomes

$$4h^3 - g_2 hf^2 - g_3 f^3 + 2gt\sqrt{f^3} + f^3(4s^3 - g_2 s - g_3)$$

$$- 4h^3 + 12hsf^2 - 8s^3 f^3.$$

Hence $$du = \frac{ds\sqrt{f^3}}{t\sqrt{f^3}}$$

$$\times\left(\frac{4s^3\sqrt{f^3} - 12hs^2\sqrt{f} + g_2 s\sqrt{f^3} + hg_2\sqrt{f} + 2g_3\sqrt{f^3} - 2gt}{-4s^3\sqrt{f^3} + 12hs^2\sqrt{f} - g_2 s\sqrt{f^3} - hg_2\sqrt{f} - 2g_3\sqrt{f^3} + 2gt}\right),$$

so that $$du = -\frac{ds}{t},$$

and taking $$s = \wp(u), \quad t = -\wp'(u),$$

we have, if* $$u = \int_{p,q}^{x,y} \frac{y\,dx - x\,dy}{\sqrt{f(x,y)}},$$

$$x = p\sqrt{f}\,\wp'(u) + \frac{\wp(u)}{2}\frac{\partial f}{\partial q} - \tfrac{1}{2}\frac{\partial h}{\partial q},$$

$$y = q\sqrt{f}\,\wp'(u) - \frac{\wp(u)}{2}\frac{\partial f}{\partial p} + \tfrac{1}{2}\frac{\partial h}{\partial p},$$

since when $u = 0$, $x/y = p/q$.

From equations (1), we can also deduce a curious formula in elliptic functions as follows:—

Take $$f(x, y) = 4x^3y - g_2xy^3 - g_3y^4,$$

so that $$h(x, y) = x^4 + \tfrac{1}{2}(g_2x^2y^2) + 2g_3xy^3 + \tfrac{1}{16}(g_2^2y^4).$$

Hence, if $4x^3 - g_2x - g_3 = f$, a perfect square when $x = p$, the general value of x satisfying the same condition is, from equations (1), putting $q = 1$ and taking quotient of both equations (1),

$$x = \frac{2pt\sqrt{f} - s(4p^3 - 3g_2p - 4g_3) + g_2p^2 + 6g_3p + \frac{1}{4}g_2^2}{2t\sqrt{f} + s(12p^2 - g_2) - 4p^3 - g_2p - 2g_3},$$

and writing $$p = \wp(u), \quad s = \wp(v),$$

we have

$$x = \frac{\left\{\begin{array}{r} 2\wp(u)\wp'(v)\wp'(v) - \wp(v)\left[4\wp^3(u) - 3g_2\wp(u) - 4g_3\right] \\ + g_2\wp^2(u) + 6g_3\wp(u) + \frac{1}{4}(g_2^2) \end{array}\right\}}{2\wp'(u)\wp'(v) + \wp(v)\left[12\wp^2(u) - g_2\right] - 4\wp^3(u) - g_2\wp(u) - 2g_3} \quad \ldots\ldots(2).$$

Now, since this solution x is expressed rationally in terms of two others, we should expect that $x = \wp(mu + nv)$ where m and n are positive or negative integers including zero, and since we find x is infinite when $u = -v$, and equal to $\wp(2u)$ when $u = v$, x should be equal to $\wp(u + v)$, which is verified thus. From (2), we should have

$$\wp(u+v) - \wp(u) = 4\wp'^2(u)\{\wp(2u) - \wp(v)\}$$

$$\div \text{ same denominator as (2)},$$

* Cf. result given in *Math. Tripos*, Part II., 1902.

since $4\wp(2u)\wp'^2(u) = 4\wp^4(u) + 2g_2\wp(u)^2 + 8g_3\wp(u) + \frac{1}{4}g_2^2$,

so that
$$\frac{4'^2\wp(u)[\wp(2u) - \wp(v)]}{\wp(u+v) - \wp(u)}$$

$$= 2\wp'(u)\wp'(v) + \wp(v)[12\wp^2(u) - g_2] - 4\wp^3(u) - g_2\wp u - 2g_3,$$

which is easily verified by examining the singularities of both sides considered as a function of v; or, by expressing both sides in terms of σ functions (noticing that the right-hand side vanishes for $v = -u, -u, 2u$).

NOTE ON A PROOF IN VALLEE-POUSSIN'S "COURS D'ANALYSE."*

By *S. Pollard.*

THE theorem proved is as follows:—

If $f(x)$ is a function which in (a, b) is uniform and bounded, and $Of(x)$ denotes the oscillation of $f(x)$ at a point x in (a, b), with the usual convention that $Of(a)$ is the oscillation to the right at a, and $Of(b)$ that to the left at b, then

$$\overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} Of(x)\,dx.$$

If δ_i represents a closed sub-interval of (a, b), we denote by M_i, m_i the upper and lower bounds of $f(x)$ in δ_i, and by Δ_i the upper bound of $Of(x)$ in δ_i.

The author asserts that it can be shewn by the usual bisection proof† that there is a mode of division of (a, b) into δ_i's where each δ_i is as small as we please, and in it

$$M_i - m_i < \Delta_i + \epsilon,$$

where ϵ is any positive number given in advance.

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* 3rd edition, t. 1, p. 254.

† As in p. 26 in No. 15 of the *Cambridge Tracts.*

Now, although the bisection proof shews the existence of a mode of division into δ_i's such that

$$M_i - m_i < \Delta_i + \epsilon,$$

it does not prove anything about the size of the δ_i's. And we cannot obtain another mode of division from this with arbitrarily small sub-intervals by further division of the δ_i's; for, as will be clear from the examples given below, we may have a part δ_κ of a δ_i, for which

$$M_\kappa - m_\kappa > \Delta_\kappa + \epsilon,$$

although $$M_i - m_i < \Delta_i + \epsilon.$$

Secondly, as Mr. Hardy has pointed out, the author states that in each δ_i,

$$\Delta_i \leq M_i + m_i,$$

and this is not necessarily true. Indeed, we can easily construct an example for which

$$\Delta_i > M_i + m_i.$$

Define $f(x)$ as follows:—

$$\begin{aligned} f(x) &= 1 \qquad (x \leq 0), \\ &= 1 - x \quad (0 \leq x \leq 1), \\ &= x - 1 \quad (1 \leq x < 2), \\ &= 0 \qquad (2 \leq x \leq 3). \end{aligned}$$

Take (0, 3) as δ_i, and (0, 1), (1, 2), (2, 3) as δ_1, δ_2, δ_3, respectively.

Then $$M_i - m_i = 1,$$

$$\Delta_i = 1,$$

and $$\Delta_i \leq M_i - m_i < \Delta_i + \epsilon.$$

But in δ_1, $$M_1 - m_1 = 1,$$

$$\Delta_1 = 0,$$

and $$M_1 - m_1 > \Delta_1 + \epsilon.$$

Again, in δ_3, $$M_3 - m_3 = 0,$$

$$\Delta_3 = 1 > M_3 - m_3.$$

A proof avoiding these defects may be given as follows:—

In virtue of Darboux's theorem in connection with definite integration*, we can divide (a, b) into sub-intervals δ_κ' such that

$$\Sigma\, \Delta'_\kappa\, \delta'_\kappa - \overline{\int_a^b} Of(x)\, dx < \epsilon \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\Sigma\, (M'_\kappa - m'_\kappa)\, \delta'_\kappa - \left\{ \overline{\int_a^b} f(x)\, dx - \underline{\int_a^b} f(x)\, dx \right\} < \epsilon \ldots\ldots (2),$$

where ϵ is any number, greater than 0, given in advance, and Δ'_κ is the upper bound of $Of(x)$ in δ'_κ.

By a known extension of Goursat's lemma† we can divide each δ'_κ into a finite number of pieces such that, if δ_i represents one of them,

$$M_i - m_i < \Delta_i + \epsilon,$$

and so $$\Sigma\, (M_i - m_i)\, \delta_i < \Sigma\, \Delta_i \delta_i + \epsilon\, (b - a) \ldots\ldots\ldots (3).$$

Since the δ_i's are obtained by subdividing the δ_κ's,

$$\overline{\int_a^b} Of(x)\, dx \leq \Sigma\, \Delta_i \delta_i \leq \Sigma\, \Delta'_\kappa\, \delta'_\kappa \ldots\ldots\ldots\ldots (4),$$

and

$$\overline{\int_a^b} f(x)\, dx - \underline{\int_a^b} f(x)\, dx \leq \Sigma\, (M_i - m_i)\, \delta_i \leq \Sigma\, (M'_\kappa - m'_\kappa)\, \delta'_\kappa \ldots (5).$$

It follows, using (5), (3), (4), (1) in order, that

$$\begin{aligned} \overline{\int_a^b} f(x)\, dx - \underline{\int_a^b} f(x)\, dx &< \Sigma\, (M_i - m_i)\, \delta_i, \\ &< \Sigma\, \Delta_i \delta_i + \epsilon\, (b - a), \\ &\leq \Sigma\, \Delta'_\kappa \delta'_\kappa + \epsilon\, (b - a), \\ &< \overline{\int_a^b} Of(x)\, dx + \epsilon + \epsilon\, (b - a). \end{aligned}$$

Since ϵ is arbitrary, it follows that

$$\overline{\int_a^b} f(x)\, dx - \underline{\int_a^b} f(x)\, dx \leq \overline{\int_a^b} Of(x)\, dx \ldots\ldots\ldots (\alpha).$$

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* See Goursat, *Cours d'Analyse*, t. I., p. 173.

† Proved as in tract 15 above.

*Again, subdivide the δ_i's into divisions η, in such a way that the total length of η's abutting on end points of δ_i's is less than $\frac{\epsilon}{M-m}$.

Then
$$\overline{\int_a^b} Of(x)\,dx = \Sigma \overline{\int_{\delta_i}} Of(x)\,dx$$
$$= \Sigma \overline{\int_{\eta}} Of(x)\,dx$$
$$= \sigma_1 + \sigma_2,$$

where σ_1, σ_2 are the sums $\Sigma \overline{\int} Of(x)\,dx$ taken respectively over η's which do and which do not abut on end points of δ_i's.

Now $Of(x) \leq M-m$ for any x in (a, b), and hence
$$\sigma_1 < \frac{\epsilon}{M-m}(M-m) = \epsilon.$$

Let η_{i1}, η_{i2}, ..., $\eta_{i\kappa}$ be the η's which lie entirely within δ_i. Then the upper bound of $Of(x)$ in each of them is not greater than $M_i - m_i$, for if x lies within δ_i,
$$Of(x) \leq M_i - m_i.$$

It follows that
$$\sigma_2 = \sum_{r=1}^{\kappa} \overline{\int_{\eta_{ir}}} Of(x)\,dx < (M_i - m_i)\,\delta_i,$$

and
$$\overline{\int_a^b} Of(x)\,dx < \Sigma (M_i - m_i)\,\delta_i + \epsilon$$
$$< \Sigma (M'_\kappa - m'_\kappa)\,\delta'^\kappa + \epsilon$$
$$< \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx + 2\epsilon,$$

using (5) and (2).

Hence,
$$\overline{\int_a^b} Of(x)\,dx \leq \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \ldots\ldots (\beta).$$

Combining (α), and (β), we have our theorem.

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* This part of the proof is due to Mr. Hardy.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy.*

XL.

Some cases of term-by-term integration of an infinite series.

1. I HAVE recently met with some interesting and apparently novel examples of failure of the process of term-by-term integration of an infinite series. The examples usually given are of a rather artificial type. A familiar one* is that in which

$$S(x) = \sum_0^\infty u_n(x) = \sum_0^\infty \{nxe^{-nx^2} - (n+1)\,xe^{-(n+1)x^2}\} = 0,$$

$$\int_0^1 S(x)\,dx = 0,$$

$$\sum_0^\infty \int_0^1 u_n(x)\,dx = \tfrac{1}{2} \sum_0^\infty \{e^{-n} - e^{-(n+1)}\} = \tfrac{1}{2}.$$

A simpler example is that in which

$$S(x) = \sum_0^\infty u_n(x) = \sum_0^\infty \{(n+1)\,x^n - (n+2)\,x^{n+1}\} = 1,$$

$$\int_0^1 S(x)\,dx = 1,$$

$$\sum_0^\infty \int_0^1 u_n(x)\,dx = \sum_0^\infty (1-1) = 0.$$

This example, however, suffers from the disadvantage that the series ceases to converge at all at one end of the range of integration; and it is in any case not less artificial than the other. That which follows is not artificial, *i.e.* constructed expressly for the purpose of illustrating theoretical difficulties. It occurred quite naturally in some analysis in which, by ignoring such difficulties, I was led momentarily to an incorrect result.

* Bromwich, *Infinite Series*, p. 118. Dr. Bromwich remarks that "it is however not easy to give a really simple example of a non-uniformly convergent series in which term-by-term integration leads to erroneous results".

2. *The series*

$$\sum_1^\infty (e^{-anx} - e^{-bn^2x^2}), \tag{1}$$

where

$$b = \tfrac{1}{4}\pi a^2, \tag{2}$$

is convergent for all positive values of x, and represents a function $\phi(x)$ continuous for all positive values of x. But it cannot be integrated term-by-term over any interval $(0, \xi)$, where $\xi > 0$.

It is evident that we may, without loss of generality, suppose

$$a = 2, \quad b = \pi.$$

The series is obviously convergent, and uniformly convergent for $x \geq \xi > 0$. Thus $\phi(x)$ is continuous for $x > 0$. Further, we have

$$\sum_1^\infty e^{-2nx} = \frac{1}{e^{2x} - 1} = \frac{1}{2x} - \frac{1}{2} + \sum_1^\infty \frac{x}{x^2 + n^2\pi^2}, \tag{3}$$

$$\sum_1^\infty e^{-\pi n^2 x^2} = \frac{1}{2x} - \frac{1}{2} + \frac{1}{x}\sum_1^\infty e^{-\pi n^2/x^2}, \tag{4}$$

and so

$$\phi(x) = x\left\{\sum_1^\infty \frac{1}{x^2 + n^2\pi^2} - \frac{1}{x^2}\sum_1^\infty e^{-\pi n^2/x^2}\right\}. \tag{5}$$

Now

$$\sum_1^\infty \frac{1}{x^2 + n^2\pi^2} \leq \frac{1}{\pi^2}\sum_1^\infty \frac{1}{n^2} = \frac{1}{6}, \tag{6}$$

and

$$\frac{1}{x^2}\sum_1^\infty e^{-\pi n^2/x^2} < \frac{1}{e\pi}\sum_1^\infty \frac{1}{n^2} = \frac{\pi}{6e}.* \tag{7}$$

Hence

$$\phi(x) = O(x) \to 0 \tag{8}$$

as $x \to 0$, and $\phi(0) = 0$, so that $\phi(x)$ is continuous for $x \geq 0$.

3. Again

$$\int_\delta^\infty \sum_1^\infty e^{-2nx}\,dx = \sum_1^\infty \int_\delta^\infty e^{-2nx}\,dx,$$

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* The function ue^{-Au}, where A and u are positive, has a maximum when $u = 1/A$, and its maximum value is $1/eA$.

since every term of the series is positive and the integral on the left-hand side convergent.* Similarly,

$$\int_\delta^\infty \sum_1^\infty e^{-\pi n^2 x}\,dx = \sum_1^\infty \int_\delta^\infty e^{-\pi n^2 x}\,dx.$$

Subtracting, we see that

$$\int_\delta^\infty \sum_1^\infty (e^{-2nx} - e^{-\pi n^2 x})\,dx = \sum_1^\infty \int_\delta^\infty (e^{-2nx} - e^{-\pi n^2 x})\,dx.$$

Thus term-by-term integration is legitimate over the range (δ, ∞). If then we can show that it is not legitimate over the range $(0, \infty)$, it will follow that it is not legitimate over the range $(0, \delta)$. But

$$\int_0^\infty (e^{-2nx} - e^{-n^2\pi x})\,dx = 0,$$

so that the result of term-by-term integration is

$$\int_0^\infty \phi(x)\,dx = 0. \tag{9}$$

We shall show that this result is certainly false, because $\phi(x)$ is positive for all positive values of x.

4. It is obvious that $\phi(x)$ is positive if $x > 2/\pi$, since then $e^{-2nx} > e^{-\pi n^2 x^2}$ for all values of n. We may therefore suppose that $x \leq 2/\pi$. We have then to use the equation (3). It is evident that

$$\frac{1}{x^2 + n^2\pi^2}$$

decreases steadily as x increases. And

$$\frac{1}{x^2} e^{-\pi n^2/x^2}$$

increases steadily as x increases from 0 to the value $n\sqrt{\pi}$, for which it has its maximum value. Also

$$n\sqrt{\pi} > 1 > 2/\pi.$$

It follows that $\phi(x)$ decreases steadily as x increases from 0 to $2/\pi$. Thus $\phi(x)$ is always positive, and the equation (9) is certainly false.

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* Bromwich, *Infinite Series*, p. 453.

5. It is easy to find the correct value of the integral (9). The integral

$$\int_0^\infty x^a \phi(x)\, dx \tag{10}$$

is uniformly convergent throughout any interval of the type $0 \leq a \leq a_0$, since

$$\phi(x) < \frac{e^{-2x}}{1-e^{-2x}} = O(e^{-2x})$$

when x is large, and

$$\phi(x) = O(x),$$

by (9), when x is small. It is therefore continuous for $a = 0$. Also

$$\int_0^\infty x^a \Sigma e^{-2nx}\, dx = 2^{-1-a}\,\Gamma(1+a)\,\Sigma n^{-1-a} = 2^{-1-a}\,\Gamma(1+a)\,\zeta(1+a),$$

and

$$\int_0^\infty x^a \Sigma e^{-\pi n^2 x^2}\, dx = \tfrac{1}{2}\pi^{-\frac{1}{2}(1+a)}\,\Gamma\{\tfrac{1}{2}(1+a)\}\,\zeta(1+a);$$

both of these term-by-term integrations being justified by the theorem appealed to in § 3. Hence

$$\int_0^\infty \phi(x)\, dx$$

$$= \lim_{a \to 0} [2^{-1-a}\,\Gamma(1+a)\,\zeta(1+a) - \tfrac{1}{2}\pi^{-\frac{1}{2}(1+a)}\,\Gamma\{\tfrac{1}{2}(1+a)\}\,\zeta(1+a)].$$

Using the approximations

$$\Gamma(1+a) = 1 - \gamma a + \ldots,$$

$$\Gamma(\tfrac{1}{2} + \tfrac{1}{2}a) = \sqrt{\pi}\,\{1 - \tfrac{1}{2}(\gamma + 2\log 2)\,a + \ldots\},$$

$$a\zeta(1+a) = 1 + \gamma a + \ldots,$$

we find that

$$\int_0^\infty \phi(x)\, dx = \tfrac{1}{4}(\log \pi - \gamma) = .14188\ldots. \tag{11}$$ *

6. The preceding example suggests one which is even simpler. Suppose that a, b, and x are positive and $a < b$. Then

$$\sum_1^\infty (ae^{-nax} - be^{-nbx}) = \frac{a}{e^{ax}-1} - \frac{b}{e^{bx}-1}.$$

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* Bromwich, *Infinite Series*, p. 351.

and

$$\int_0^\infty \sum_1^\infty (ae^{-nax} - be^{-nbx})\,dx = \log\frac{b}{a}. \tag{12}$$

But

$$\sum_1^\infty \int_0^\infty (ae^{-nax} - be^{-nbx})\,dx = \sum_1^\infty \left(\frac{1}{n} - \frac{1}{n}\right) = 0. \tag{13}$$

It is easily verified in this case also that the sum of the series is always positive, so that (13) is certainly false. This example has, however, the same theoretical disadvantage as the second of those quoted in § 1: the series is not convergent at all for $x = 0$.

On the other hand it has the advantage that the failure of term-by-term integration over an arbitrarily small interval $(0, \delta)$ can be recognised directly: for

$$\Sigma \int_0^\delta (ae^{-nax} - be^{-nbx})\,dx = -\Sigma\left(\frac{e^{-na\delta} - e^{-nb\delta}}{n}\right) = \log\left(\frac{1-e^{-a\delta}}{1-e^{-b\delta}}\right),$$

$$\Sigma \int_0^\delta \left(\frac{a}{e^{ax}-1} - \frac{b}{e^{bx}-1}\right)dx = \log\left(\frac{1-e^{-a\delta}}{1-e^{-b\delta}}\right) - \log\left(\frac{a}{b}\right).$$

THE NINE-EIGHTEEN SECTION OF THE ZETA FUNCTION.

By *J. R. Wilton, M.A., D.Sc.*

1. The present communication carries on the work undertaken in "The Seven-Fourteen Section of the Zeta Function,"* which will be referred to as the *Fourteen Section*. The notation there used is followed here, and the numbering of the equations is consecutive to that of the equations of the *Fourteen Section*.

The eighteen-section.

In this case the dynamical application of the results given below requires the evaluation of the pseudo-elliptic integral, in the circular form, when the parameter v is equal to $K + fK'i$, where

$$f = \frac{2k+1}{18} \quad (k = 1, 2, \ldots, 8).$$

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* pp. 129–138 of the present volume.

The value of the integral is

$$I(v) = \int_0^x \left(\frac{Q}{D-x^2} - P\right) \frac{dx}{\sqrt{(X_1 X_2)}}$$

$$= \frac{1}{9} \cos^{-1} \frac{R_0 + R_1 x + R_2 x^2 + \ldots + R_7 x^7 + x^8}{(D-x^2)^{\frac{9}{2}}} \sqrt{(\tfrac{1}{2} X_1)} - \frac{\frac{1}{4}\pi}{9}$$

$$= \frac{1}{9} \sin^{-1} \frac{R_0 - R_1 x + R_2 x^2 - \ldots - R_7 x^7 + x^8}{(D-x^2)^{\frac{9}{2}}} \sqrt{(\tfrac{1}{2} X_2)} - \frac{\frac{1}{4}\pi}{9},$$

where X_1, X_2, P, Q, and D are defined by precisely the same equations as in § 1 of the *Fourteen Section*.

But in the present investigation $P = P(v)/P(9v)$, and the leading coefficient in descending powers of x is

$$R_7 = 9P - 1 = N\{9P(v) - P(9v)\}.$$

The statement of the actual reduction required in this connection is made in § 4 of the *Fourteen Section*, equations (21) to (27). The algebra, though a little heavy, is much less tedious than might have been expected.

As in the former case, we consider separately the two terms in equation (21),

$$K \equiv \frac{N(y-x-y^2)^3}{y^3(y-x-y^2) - (y-x)^3} \ldots\ldots\ldots\ldots (28),$$

and

$$3N = \tfrac{3}{2} \frac{(1-a)(a-b)^3}{a^3 e (1+b)} \ldots\ldots\ldots\ldots (29),$$

where e, $= 1/(1+c)$, is used for simplicity.

We have

$$a^3 - b^4 = a^3(1-a)e^2 \ldots\ldots\ldots\ldots (8),$$

and substituting this value of b in (22), we find, after reduction,

$$a^3 e^3 - (1+2a)e^2 + (1+a+a^2)e - 1 = 0 \ldots\ldots (30).$$

By means of (8) and (30) we easily reduce $3N$ to the form required. We have

$$\frac{(1-a)(a-b)^3}{a^3 e (1+b)}$$

$$= \frac{(1-a)\left[(1+a)^3 - 3(1+a)^2(1+b) + 3(1+a)(1+b)^2 - (1+b)^3\right]}{a^3 e (1+b)}$$

$$= \frac{(1+a)^3}{a^3 e} \frac{1-a}{1-b^4}(1-b+b^2-b^3) - \frac{1-a}{a^3 e}\{1+3a+3a^2-(1+3a)b+b^2\}.$$

But $$\frac{(1+a)^3}{a^3e}\,\frac{1-a}{1-b^4}=\frac{(1+a)^3}{a^3e\,(1+a+a^2+a^3e^2)},$$

from (8). And since, from (30),

$$(1+a)^3=(1+a)^3-a^3\{a^3e^3-(1+2a)\,e^2+(1+a+a^2)\,e-1\}$$
$$=(1+2a-a^3e)\,(1+a+a^2+a^3e^2),$$

we have

$$2N=\left(\frac{1+2a}{a^3e}-1\right)(1-b+b^2-b^3)$$
$$-\frac{1-a}{a^3e}\{1+3a+3a^2-(1+3a)\,b+b^2\},$$

i.e.,

$$3N=-\frac{3}{2}+\frac{9}{2e}+\left(\frac{3}{2}-\frac{9}{2ae}\right)b$$
$$-\left(\frac{3}{2}-\frac{9}{2a^2e}\right)b^2+\left(\frac{3}{2}-\frac{3+6a}{2a^3e}\right)b^3\ldots(31).$$

2. We return now to the calculation of K as defined in (28). Using the values of x and y in terms of m and α given in equations (2) of the *Fourteen Section*, we find, after a rather heavy but perfectly straightforward reduction,

$$K=-\frac{N\epsilon^3}{E},$$

where

$$E=-1+\frac{\beta}{1-m}(7-10\beta+4\beta^2)+\frac{2\beta\,(\beta-1)\,(2-\beta)}{(1-m)^2}+\frac{\beta\,(\beta-1)^2}{(1-m)^3},$$

$$\epsilon=2\beta-1,\quad\text{and}\quad\beta=\frac{(1-2m)\,\alpha}{m-m^2}.$$

[For the notation, see *Phil. Trans.*, p. 285, equations (23) and (24).]

We proceed to express K/N in terms of the quantities p, $=1-2m$, and ϵ, and we find, after a slight reduction,

$$E=F/(1+p)^3,$$

where $$F=\epsilon^3+\epsilon p\,(\epsilon^2-\epsilon+1)+\epsilon^2p^2(\epsilon-2)-p^3$$
$$=[\epsilon^3+\epsilon^2p^2(\epsilon-2)]\,(1-1/b)$$

by equation (3), p. 292, of *Phil. Trans.**

* A similar artifice might have been used in the case of the fourteen section, using equation (6), p. 290, of *Phil. Trans.*, but I find that it does not appreciably shorten the work as previously given.

Thus $$K=\frac{N\epsilon b\,(1+p)^3}{(1-b)\,\{\epsilon\,(1+p^2)-2p^2\}}.$$

Now [*Phil. Trans.*, p. 285, equations (19) and (24)],

$$\epsilon=\frac{2a^3+a^4-(1+2a)\,b^4}{a\,(1-b^2)\,(a^2+b^2)},$$

$$p=\frac{a-b^2}{b\,(1-a)},\qquad 1+p=\frac{(a+b)\,(1-b)}{b\,(1-a)};$$

therefore

$$\epsilon\,(1+p^2)-2p^2=\epsilon\,(1+p)^2-2p\,(\epsilon+p)$$
$$=\frac{(a^2-b^2)^3\,(1+b^2)}{ab^2\,(1-a)^2\,(1-b^2)\,(a^2+b^2)},$$

after a straightforward reduction. Hence

$$K=\frac{(1-b)^3\{2a^3+a^4-(1+2a)\,b^4\}}{2a^3e\,(1-b^4)}.$$

But, using (8) and (30), we have

$$(1+2a-a^3e)\,b^4=a^3(1+2a-a^3e)\,(1-e^2+ae^2)$$
$$=a^3\,(2+a-e),$$

so that $$2a^3+a^4-(1+2a)\,b^4=a^3e\,(1-b^4);$$

and therefore $$K=\tfrac{1}{2}\,(1-b)^3 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (32).$$

Adding the results (31) and (32), we find

$$N\{9P(v)-P(9v)\}=3N+K$$
$$=-1+\frac{9}{2a^2e}\,(a^2-ab+b^2)+\left(1-\frac{3+6a}{2a^3e}\right)b^3 \qquad \ldots\ldots (33).$$

We still require to express the last term on the right of (33) in the form (a function of $r)/b$, where r is defined in equation (24) of the *Fourteen Section*. To do this, we have, from (8) and (30),

$$\left(1-\frac{3+6a}{2a^3e}\right)b^3=\frac{1}{b}\left\{-\frac{5+4a}{2e}+1+\tfrac{1}{2}\,(1-a)\,(1+2a)\,e\right\} \ldots (34).$$

Putting $1+a=c/r=(1-e)/er$ in equation (30), we obtain

$$(r+1)^3e^2+(r^3+r^2-1)\,e-r=0;$$

and therefore $$e = -\frac{r^3 + r^2 - 1 - \sqrt{R}}{2(r+1)^3},$$

where R is defined in (26). And after a rather long reduction, using (34), we find

$$N\{9P(v) - P(9v)\} = -1 + \frac{9}{2a^2e}(a^2 - ab + b^2) - \frac{\theta + \phi\sqrt{R}}{4r^3(r+1)^3 b} \quad \ldots\ldots(35),$$

where

$$\theta = 4r^9 + 21 + 48 + 66 + 66 + 57 + 45 + 30 + 12r + 2,$$

$$\phi = 4r^6 + 17 + 23 + 7 - 8 - 8r - 2.$$

To connect this result with that for the nine section of the zeta function, we make precisely the same quadric transformation as was made when we wished to connect the result for the fourteen section with that for the seven section, except that in this case the fraction f is equal to

$$\frac{2k+1}{18} \quad (k = 1, 2, \ldots, 8),$$

and the last relation in § 2 of the *Fourteen Section* must be changed* to

$$\frac{9\,\mathrm{zs}\,2fG'}{\gamma'} = \left(1 - \frac{3+6a}{2a^3e}\right) b^3 - 9\,\frac{1+3a}{4b}\,e + 9\,\frac{1+a}{4ab}$$

$$= \frac{r^9 + 3 + 3 - 6 - 42 - 69 - 9 + 57 + 30r + 5 + (r^6 + 2 - 1 + 4 - 2 - 20r - 5)\sqrt{R}}{4r^3(r+1)^3 b}.$$

The twenty-section.

3. The corresponding reduction in the case of the twenty section is straightforward and, considering the complication of the formulæ, not very heavy.

An error seems to have crept into equation (16) on p. 221 of *Phil. Trans.*, and I find

$$N\{10P(v) - P(10v)\} = N\left\{\frac{5}{2}(1+y) + \frac{N_{10}}{\gamma_{10}}\right\} \quad \ldots(36),$$

where

$$\gamma_{10} = (y-x)\{y^2[(y-x)x - y^3] - x(y-x-y^2)^2\},$$

$$N_{10} = -y^3\{5x(y-x)^2 - 2y^2(y^2-x^2) - y^2(y-x-y^2)\}.$$

* This relation is supplied by Sir George Greenhill (*Archiv. der Math.*, 1900, p. 74).

Putting $m - m^2 = n,$

and using the function β introduced in § 2, $\beta = p\alpha/n$, we find, on reduction,

$$\gamma_{10} = -\frac{\alpha^4 n^4}{(\alpha - m)^8}\{(1-4n)\beta^4 - 2(1-5n)\beta^3 + (1-6n-4n^2)\beta^2 + 6n^2\beta - n^2\}$$

$$= -\frac{\alpha^4 n^4}{(\alpha - m)^8}(\beta^2 - \beta + n)\{(1-4n)\beta^2 - (1-6n)\beta - n\},$$

$$N_{10} = \frac{m\alpha^5\beta n^3(1-4n)}{(\alpha - m)^9}\{2n\beta^3 - (1+2n)\beta^2 + 2(1-2n+2n^2)\beta - (1-4n+5n^2)\}.$$

And, as in the sixteen section,

$$N = \frac{1}{\sqrt{C_0}} = \frac{m-\alpha}{m\alpha\alpha}.$$

Therefore, from (36),

$$N\{10P(v) - P(10v)\} = \tfrac{5}{2}N(1+y) + \frac{NN_{10}}{\gamma_{10}}$$

$$= \frac{5}{2a\alpha} - \frac{5}{a} + \frac{\beta(1-4n)}{an(\beta^2 - \beta + n)} \times \frac{2n\beta^3 - (1+2n)\beta^2 + 2(1-2n+2n^2)\beta - (1-4n+5n^2)}{(1-4n)\beta^2 - (1-6n)\beta - n}.$$

What is required is the value of this expression in terms of a parameter z, which is connected with α, β, a, and n by the equations

$$\beta = \frac{1}{1+b}, \quad n = \frac{b(1+b)}{E} \quad \ldots\ldots\ldots\ldots\ldots\ldots (37),$$

where

$$E = (1+2b)^2 - a^4,$$

$$\frac{1}{\alpha^2} = \frac{E(1-a^4)}{b^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (38)$$

[*Phil. Trans.*, p. 295, equations (18), (21), and (24)];

$$b = ac,$$

$$c^6 + 3(t+1)c^4 - (t^3 + t^2 - 2t - 3)c^2 - t^2 - t + 1 = 0 \ldots (39),$$

where $$t = a + \frac{1}{a};$$

$$ty^3 + 3ty^2 - (t^2 + t + 4)\,y + t = 0 \quad \ldots\ldots\ldots\ldots (40),$$

where $$c^2 = yt - 1;$$

$$y = \frac{1-z}{1+z} \quad \ldots\ldots\ldots\ldots\ldots\ldots (41),$$

$$t - 2 = \frac{4\sqrt{Z}\{\sqrt{Z}+1\}}{(1-z)(1+z)^2}, \quad t + 2 = \frac{4\{\sqrt{Z}+1\}}{(1-z)(1+z)^2} \ldots (42),$$

where $$Z = -z + z^2 + z^3;$$

$$c^2 = \frac{1 - 5z - z^2 + z^3 + 4\sqrt{Z}}{(1+z)^3} \quad \ldots\ldots\ldots\ldots (43),$$

$$\left(\frac{1}{a} - a\right)^2 = t^2 - 4 = \frac{16\sqrt{Z}\{\sqrt{Z}+1\}^2}{(1-z)^2(1+z)^4} \quad \ldots\ldots\ldots (44).$$

[*Phil. Trans.*, p. 303, equations (9) to (19)].

Proceeding with the reduction we find

$$\beta^2 - \beta + n = \frac{b}{(1+b)^2}\,\frac{a^4 - b^3 - b^2 - b}{E},$$

$$(1-4n)\beta^2 - (1-6n)\beta - n = \frac{b}{(1+b)^2}\,\frac{a^4 - b^3 - b^2 + b}{E},$$

$$\begin{aligned}(1+b)^2E^2\{2n\beta^3 - (1+2n)\beta^2 + 2(1-2n+2n^2)\beta - (1-4n+5n^2)\} \\ = b^2\{-E^2 + 2E(1+4b+2b^2) - (1+b)^3(1+5b)\} \\ = b^2(2b^2 - 5b^4 + 4b^2a^4 - a^8).\end{aligned}$$

Let $$K \equiv N\{10P(v) - P(10v)\} - \frac{5}{2a\alpha} + \frac{5}{a}$$

$$= \frac{1-a^4}{ab}\,\frac{2b^2 - 5b^4 + 4b^2a^4 - a^8}{(a^4 - b^2)^2 - b^2(1-b^2)^2}.$$

Using equation (39) we find

$$\begin{aligned}-a^4c^6 + 3a^2c^4 - (1+2a^4)\,c^2 + a^6 \\ = (1 + a + a^2 + a^3)\{3a^2c^4 - (1+a^4)\,c^2 - a^2\}.\end{aligned}$$

Whence $$K\frac{a^2c}{a-1} - 1 = \frac{(a^2 - c^2)\{(1/a^2) + a^2 - 2c^2\}}{3c^4 - \{(1/a^2) + a^2\}\,c^2 - 1}$$

$$= \frac{(a^2 - c^2)(t - 2y)}{(3y^2+1)\,t - (t^2+4)\,y}.$$

But, from (40),

$$(3y^2+1)t-(t^2+4)y=ty(1-y^2);$$

and, using the relations (41) and (42), we have

$$t-2y=\frac{4(z^2+\sqrt{Z})}{(1-z)(1+z)^2}.$$

Whence

$$K=-\frac{1-a}{a^2c}\left\{1+\frac{(a^2-c^2)(z^2+\sqrt{Z})(1+z)^3}{2z(1-z)(1+\sqrt{Z})^2}\right\}.$$

From (44), taking $a>1$, we find

$$a=\frac{(1+\sqrt{Z})(1+\sqrt[4]{Z})^2}{(1-z)(1+z)^2};$$

and, using (43),

$$(a^2-c^2)(1-\sqrt{Z})^2=4\{z^2+\sqrt{Z}+\sqrt[4]{Z}(1+\sqrt{Z})\}.$$

Finally,

$$K=\frac{2\{-2+3z+z^2-z^3+(5+2z)\sqrt[4]{Z}-5\sqrt{Z}+5\sqrt[4]{Z^3}\}}{c(1-\sqrt{Z})^2}$$

and the expression for

$$N\{10P(v)-P(10v)\}=K+\frac{5}{2a\alpha}-\frac{5}{a}$$

in terms of z, which is extremely complicated, can be written down by means of the equations (38), (43), and (44).

4. The only remaining section which can at present be completely worked out is the twenty-two section, and there is still a difficulty to be overcome before the final expression can be given in terms of a single parameter.

I have, however, obtained the expression of

$$N\{11P(v)-P(11v)\}$$

in terms of the a, b, and e already used in the fourteen and eighteen sections. But its exposition, together with certain numerical verifications of the results already given, is reserved for a further communication.

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SUMMATION OF A CERTAIN SERIES.

By *S. Ramanujan.*

1. Let

$$\Phi(s, x) = \sum_{n=0}^{n=\infty} \{\sqrt{(x+n)} + \sqrt{(x+n+1)}\}^{-s}$$
$$= \sum_{n=0}^{n=\infty} \{\sqrt{(x+n+1)} - \sqrt{(x+n)}\}^{s}.$$

The object of this paper is to give a finite expression of $\Phi(s, 0)$ in terms of Riemann ζ-functions, when s is an odd integer greater than 1.

Let $\zeta(s, x)$, where $x > 0$, denote the function expressed by the series

$$x^{-s} + (x+1)^{-s} + (x+2)^{-s} + \ldots,$$

and its analytical continuations. Then

$$(1) \qquad \zeta(s, 1) = \zeta(s), \quad \zeta(s, \tfrac{1}{2}) = (2^s - 1)\,\zeta(s),$$

where $\zeta(s)$ is the Riemann ζ-function;

$$(2) \qquad \zeta(s, x) - \zeta(s, x+1) = x^{-s};$$

$$(3) \quad \begin{cases} 1^s + 2^s + 3^s + \ldots + n^s = \zeta(-s) - \zeta(-s, n+1) \\ 1^s + 3^s + 5^s + \ldots + (2n-1)^s = (1-2^s)\,\zeta(-s) - 2^s\,\zeta(-s, n+\tfrac{1}{2}), \end{cases}$$

if n is a positive integer; and

$$(4) \quad \lim_{x \to \infty} \left\{ \zeta(s, x) - \tfrac{1}{2}x^{-s} + \left(\frac{x^{1-s}}{1-s} - B_2 \frac{s}{2!} x^{-s-1} + B_4 \frac{s(s+1)(s+2)}{4!} x^{-s-3} \right.\right.$$
$$\left.\left. - B_6 \frac{s(s+1)(s+2)(s+3)(s+4)}{6!} x^{-s-5} + \ldots \text{ to } n \text{ terms} \right) \right\} = 0,$$

if n is a positive integer, $-(2n-1) < s < 1$, and $B_2 = \frac{1}{6}$, $B_4 = \frac{1}{30}$, $B_6 = \frac{1}{42}$, $B_8 = \frac{1}{30}$, ..., are Bernoulli's numbers.

Suppose now that

$$\psi(x) = 6\zeta(-\tfrac{1}{2}, x) + \sqrt{x}(4x-3) + \Phi(3, x).$$

Then from (2) we see that

$$\psi(x) - \psi(x+1) = 6\sqrt{x} + \sqrt{x}\,(4x-3) - \sqrt{(x+1)}\,(4x+1)$$
$$+ \{\sqrt{(x+1)} - \sqrt{x}\}^3 = 0;$$

and from (4) that $\psi(x) \to 0$ as $x \to \infty$. It follows that $\psi(x) = 0$. That is to say,

$$(5) \qquad 6\zeta(-\tfrac{1}{2}, x) + \sqrt{x}\,(4x-3) + \Phi(3, x) = 0.$$

Similarly, we can show that

$$(6) \quad 40\zeta(-\tfrac{3}{2}, x) + \sqrt{x}\,(16x^2 - 20x + 5) + \Phi(5, x) = 0.$$

2. Remembering the functional equation satisfied by $\zeta(s)$, viz.,

$$(7) \qquad \zeta(1-s) = 2(2\pi)^{-s}\,\Gamma(s)\,\zeta(s)\cos\tfrac{1}{2}\pi s,$$

we see from (3) and (5) that

$$(8) \quad \sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{n} = \tfrac{2}{3}n^{\frac{3}{2}} + \tfrac{1}{2}\sqrt{n} - \frac{1}{4\pi}\zeta(\tfrac{3}{2}) + \tfrac{1}{6}\Phi(3, n);$$

and

$$(9) \quad \sqrt{1} + \sqrt{3} + \sqrt{5} + \ldots + \sqrt{(2n-1)} = \tfrac{1}{3}(2n-1)^{\frac{3}{2}} + \tfrac{1}{2}\sqrt{(2n-1)} + \frac{\sqrt{2}-1}{4\pi}\zeta(\tfrac{3}{2}) + \frac{1}{3\sqrt{2}}\Phi(3, n-\tfrac{1}{2}).$$

Similarly from (6), we have

$$(10) \quad 1\sqrt{1} + 2\sqrt{2} + 3\sqrt{3} + \ldots + n\sqrt{n} = \tfrac{2}{5}n^{\frac{5}{2}} + \tfrac{1}{2}n^{\frac{3}{2}} + \tfrac{1}{8}\sqrt{n} - \frac{3}{16\pi^2}\zeta(\tfrac{5}{2}) + \tfrac{1}{40}\Phi(5, n);$$

and

$$(11) \quad 1\sqrt{1} + 3\sqrt{3} + 5\sqrt{5} + \ldots + (2n-1)\sqrt{(2n-1)} = \tfrac{1}{5}(2n-1)^{\frac{5}{2}} + \tfrac{1}{2}(2n-1)^{\frac{3}{2}} + \tfrac{1}{4}\sqrt{(2n-1)} + \frac{3(2\sqrt{2}-1)}{16\pi^2}\zeta(\tfrac{5}{2}) + \frac{1}{10\sqrt{2}}\Phi(5, n-\tfrac{1}{2}).$$

It also follows from (5) and (6) that

$$(12) \quad \sqrt{(a+d)} + \sqrt{(a+2d)} + \sqrt{(a+3d)} + \ldots + \sqrt{(a+nd)} = C + \frac{2}{3d}(a+nd)^{\frac{3}{2}} + \tfrac{1}{2}\sqrt{(a+nd)} + \tfrac{1}{6}\sqrt{d}\,\Phi(3, n+a/d);$$

and

$$(13) \quad (a+d)^{\frac{3}{2}} + (a+2d)^{\frac{3}{2}} + (a+3d)^{\frac{3}{2}} + \ldots + (a+nd)^{\frac{3}{2}} = C' + \frac{2}{5d}(a+nd)^{\frac{5}{2}} + \tfrac{1}{2}(a+nd)^{\frac{3}{2}} + \tfrac{1}{8}d\sqrt{(a+nd)} + \tfrac{1}{40}d\sqrt{d}\,\Phi(5, n+a/d),$$

where C and C' are independent of n.

Putting $n=1$ in (8) and (10), we obtain

$$(14)\quad \Phi(3,0)=\frac{3}{2\pi}\zeta(\tfrac{3}{2}),\qquad \Phi(5,0)=\frac{15}{2\pi^2}\zeta(\tfrac{5}{2}).$$

3. The preceding results may be generalised as follows. If s be an odd integer greater than 1, then

$$(15)\quad \Phi(s,x)+\tfrac{1}{2}\{\sqrt{x}+\sqrt{(x-1)}\}^s+\tfrac{1}{2}\{\sqrt{x}-\sqrt{(x-1)}\}^s$$
$$+\frac{s}{1!}2^{s-2}\zeta(1-\tfrac{1}{2}s,x)+\frac{s(s-4)(s-5)}{3!}2^{s-6}\zeta(3-\tfrac{1}{2}s,x)$$
$$+\frac{s(s-6)(s-7)(s-8)(s-9)}{5!}2^{s-10}\zeta(5-\tfrac{1}{2}s,x)$$
$$+\frac{s(s-8)(s-9)(s-10)(s-11)(s-12)(s-13)}{7!}2^{s-14}$$
$$\times\zeta(7-\tfrac{1}{2}s,x)+\ldots\text{to }[\tfrac{1}{4}(s+1)]\text{ terms}=0,$$

where $[x]$ denotes, as usual, the integral part of x. This can be proved by induction, using the formula

$$(16)\quad \{\sqrt{x}+\sqrt{(x\pm1)}\}^s+\{\sqrt{x}-\sqrt{(x\pm1)}\}^s$$
$$=(2\sqrt{x})^s\pm\frac{s}{1!}(2\sqrt{x})^{s-2}+\frac{s(s-3)}{2!}(2\sqrt{x})^{s-4}$$
$$\pm\frac{s(s-4)(s-5)}{3!}(2\sqrt{x})^{s-6}+\ldots\text{to }[1+\tfrac{1}{2}s]\text{ terms},$$

which is true for all positive integral values of s.

Similarly, we can show that if s is a positive even integer, then

$$(17)\quad \frac{s}{1!}2^{s-2}\{\zeta(1-\tfrac{1}{2}s)-\zeta(1-\tfrac{1}{2}s,x)\}$$
$$+\frac{s(s-4)(s-5)}{3!}2^{s-6}\{\zeta(3-\tfrac{1}{2}s)-\zeta(3-\tfrac{1}{2}s,x)\}$$
$$+\frac{s(s-6)(s-7)(s-8)(s-9)}{5!}2^{s-10}\{\zeta(5-\tfrac{1}{2}s)-\zeta(5-\tfrac{1}{2}s,x)\}$$
$$+\ldots\text{to }[\tfrac{1}{4}(s+2)]\text{ terms}$$
$$=\tfrac{1}{2}\{\sqrt{x}+\sqrt{(x-1)}\}^s+\tfrac{1}{2}\{\sqrt{x}-\sqrt{(x-1)}\}^s-1.$$

Now, remembering (7) and putting $x=1$ in (15), we obtain

$$(18)\quad \Phi(s,0) = -\frac{s}{\sqrt{2}}\pi^{-\frac{1}{2}(1+s)}\cos\tfrac{1}{4}\pi s\left\{1.3.5.\ldots s-2.\pi\,\zeta(\tfrac{1}{2}s)\right.$$

$$-3.5.7\ldots s-4.\tfrac{1}{2}(s-5).\tfrac{1}{3}\pi^3\,\zeta(\tfrac{1}{2}s-2)$$

$$+5.7.9\ldots s-6.\tfrac{1}{2}(s-7).\tfrac{1}{4}(s-9).\tfrac{1}{5}\pi^5\,\zeta(\tfrac{1}{2}s-4)$$

$$-7.9.11\ldots s-8.\tfrac{1}{2}(s-9).\tfrac{1}{4}(s-11).\tfrac{1}{6}(s-13).\tfrac{1}{7}\pi^7\,\zeta(\tfrac{1}{2}s-6)$$

$$+9.11.13\ldots s-10.\tfrac{1}{2}(s-11).\tfrac{1}{4}(s-13).\tfrac{1}{6}(s-15).\tfrac{1}{8}(s-17)$$

$$\left..\tfrac{1}{9}\pi^9\,\zeta(\tfrac{1}{2}s-8)-\ldots\text{ to }\left[\tfrac{1}{4}(s+1)\right]\text{ terms}\right\},$$

if s is an odd integer greater than 1. Similarly, putting $x=\frac{1}{2}$ in (15), we can express $\Phi(s,\frac{1}{2})$ in terms of ζ-functions, if s is an odd integer greater than 1.

4. It is also easy to show that, if

$$\Psi(s,x) = \sum_{n=0}^{n=\infty}\frac{\{\sqrt{(x+n)}+\sqrt{(x+n+1)}\}^{-s}}{\sqrt{\{(x+n)(x+n+1)\}}},$$

then

$$(19)\quad \Psi(s,x)-\tfrac{1}{2}\frac{\{\sqrt{x}+\sqrt{(x-1)}\}^s-\{\sqrt{x}-\sqrt{(x-1)}\}^s}{\sqrt{\{x(x-1)\}}}$$

$$=\frac{s-2}{1!}2^{s-2}\,\zeta(2-\tfrac{1}{2}s,\,x)+\frac{(s-4)(s-5)(s-6)}{3!}2^{s-6}\,\zeta(4-\tfrac{1}{2}s,\,x)$$

$$+\frac{(s-6)(s-7)(s-8)(s-9)(s-10)}{5!}2^{s-10}\,\zeta(6-\tfrac{1}{2}s,\,x)$$

$$+\ldots\text{to }\left[\tfrac{1}{4}(s+1)\right]\text{ terms},$$

provided that s is a positive odd integer. For example,

$$(20)\quad \begin{cases}\Psi(1,x)=\dfrac{1}{\sqrt{x}},\\ \Psi(3,x)=4\sqrt{x}-\dfrac{1}{\sqrt{x}}+2\zeta(\tfrac{1}{2},x),\\ \Psi(5,x)=16x\sqrt{x}-12\sqrt{x}+\dfrac{1}{\sqrt{x}}+24\zeta(-\tfrac{1}{2},x),\end{cases}$$

and so on.

THE ORTHOCENTRIC $(n+2)$-ANGLE.

By *T. C. Lewis, M.A.*

THE following shorter and more elementary proofs may conveniently be substituted for paragraphs numbered 3, 4, 5, 6 in the paper on "The Orthocentric $(n+2)$-angle" published in the *Messenger of Mathematics*, vol. xliv., p. 27. They are numbered to take their appropriate place in that paper.*

3. We have

$$\Sigma \frac{(x-a_k)\{(x-a_k)^2+(y-b_k)^2+\ldots-\rho_k^2\}}{\rho_k^2}=0,$$

therefore $$2x+\Sigma\frac{a_k x_k}{\rho_k}=0.$$

Similarly, $$2y+\Sigma\frac{b_k x_k}{\rho_k}=0, \text{ \&c., \&c.}$$

Thus the Cartesian coordinates are determined in terms of the Darboux coordinates.

4. Also $$\Sigma\frac{\{(x-a_k)^2+(y-b_k)^2+\ldots-\rho_k^2\}^2}{\rho_k^2}=\Sigma x_k^2$$
$$=0,$$

therefore $$\Sigma\{(x-a_k)^2+(y-b_k)^2+\ldots-\rho_k^2\}\frac{x_k}{\rho_k}=0,$$

therefore $$-2(x^2+y^2+\ldots)+4(x^2+y^2+\ldots)+\Sigma x_k x_k'=0,$$

where x_1', x_2', ... are the coordinates of the origin of the Cartesian coordinates.

Therefore $$2\rho^2+\Sigma x_k x_k'=0,$$

where ρ is the length of the line joining the points $(x_1, x_2, \ldots)$ and $(x_1', x_2', \ldots)$.

This may also be written in the form

$$\Sigma(x_k-x_k')^2=4\rho^2 \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

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* There is a misprint in the fifth line of §14, p. 36, which should read
$\Sigma a_k x_k=0.$

Also, the number '7' prefixed to the seventh paragraph on p. 30 was broken off in the printing.

5. Thus $$\frac{S}{\rho} = \frac{(x-a)^2+(y-b)^2+\ldots-\rho^2}{\rho}$$
$$= \frac{-\frac{1}{2}\Sigma x_k x_k' + \frac{1}{2}\rho^2 \Sigma (x_k/\rho_k)}{\rho},$$

where $(x_1, x_2, \ldots)$, $(x_1', x_2', \ldots)$ are the points $(x, y, \ldots)$, $(a, b, \ldots)$, respectively, the latter being the centre of the n-sphere $S=0$. Therefore

$$\frac{S}{\rho} = \tfrac{1}{2}\left(\frac{\rho}{\rho_1} - \frac{x_1'}{\rho}\right) x_1 + \ldots$$
$$= \frac{\rho^2 - \rho_1 x_1'}{2\rho\rho_1} x_1 + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots (4),$$
$$= \alpha_1 x_1 + \alpha_2 x_2 + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

where α_k is the cosine of the angle of intersection of the n-spheres S and $x_k = 0$. The equation of S is therefore

$$\Sigma \alpha_k x_k = 0.$$

6. It follows at once that

$$\Sigma \frac{\alpha_k}{\rho_k} = \frac{1}{\rho} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6),$$
$$\Sigma \alpha_k^2 = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7),$$

$\Sigma \alpha_k \beta_k =$ cosine of the angle of intersection of

$$\Sigma \alpha_k x_k = 0 \text{ and } \Sigma \beta_k x_k = 0 \ldots (8),$$

$$x_k' = \frac{\rho(\rho - 2\alpha_k \rho_k)}{\rho_k} \ldots\ldots\ldots\ldots\ldots\ldots (9).$$

And the expression on the right-hand side of (4) equated to zero is the equation of an n-sphere of radius ρ and having its centre at the point $(x_1', x_2', \ldots)$.

The numbering of the results from (3) to (9) has necessarily been altered, as the order of their appearance is changed.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy.*

XLI.

On the convergence of certain integrals and series.

1. THEOREM 1. *Let $f(x)$ be any positive summable function and*

$$F(x) = \int_a^x f(t)\,dt \quad (a>0).$$

Then the convergence of any one of the three integrals

$$(1)\ \int_a^\infty \frac{fF}{x}\,dx, \quad (2)\ \int_a^\infty \left(\frac{F}{x}\right)^2 dx, \quad (3)\ \int_a^\infty\int_a^\infty \frac{f(x)\,f(y)}{x+y}\,dx\,dy$$

implies that of the others.

(i) We have

$$\int_a^x \left(\frac{F}{t}\right)^2 dt = 2\int_a^x \frac{fF}{t}\,dt - \frac{\{F(x)\}^2}{x},$$

from which it follows at once that the convergence of (1) implies that of (2). On the other hand, if (2) is convergent, we have

$$\frac{\{F(x)\}^2}{x} = x\left\{\frac{F(x)}{x}\right\}^2 \leq 4\int_x^{2x}\left(\frac{F}{t}\right)^2 dt = o(1),$$

$$2\int_a^x \frac{fF}{t}\,dt = \int_a^x \left(\frac{F}{t}\right)^2 dt + o(1);$$

and so (1) is convergent. Thus the convergence of either (1) or (2) involves that of the other.

(ii) It is evident that (3) will converge if and only if

$$\int_a^\infty f(x)\,dx \int_x^\infty \frac{f(y)\,dy}{x+y}$$

converges. But

$$\frac{1}{y} > \frac{1}{x+y} > \frac{1}{2y},$$

if $y>x>a$. It follows that (3) will converge if and only if

$$\int_a^\infty f(x)\,dx \int_x^\infty \frac{f(y)}{y}\,dy$$

converges, *i.e.* if and only if

$$\int_a^\infty \frac{f(y)}{y}\,dy \int_a^y f(x)\,dx = \int_a^\infty \frac{fF}{y}\,dy$$

converges. Thus the convergence of either (1) or (3) involves that of the other.

2. There is an analogous theorem for series, viz.:

THEOREM 2. *If a_n is positive, and*

$$A_n = a_1 + a_2 + \ldots + a_n,$$

then the convergence of any one of the three series

$$(1)\ \Sigma \frac{a_n A_n}{n},\quad (2)\ \Sigma \left(\frac{A_n}{n}\right)^2,\quad (3)\ \Sigma\Sigma \frac{a_m a_n}{m+n},$$

implies that of the others.

The proof of this theorem is so much like that of Theorem 1 that it need hardly be inserted here.

3. A well-known theorem of Hilbert asserts that the convergence of

$$\int_a^\infty f^2\,dx$$

implies that of the double integral of Theorem 1, and the convergence of

$$\Sigma a_n^2$$

that of the double series of Theorem 2. Combining these results with Theorems 1 and 2, we obtain

THEOREM 3. *The convergence of*

$$\int_a^\infty f^2\,dx$$

implies that of

$$\int_a^\infty \left(\frac{F}{x}\right)^2 dx.$$

THEOREM 4. *The convergence of*

$$\Sigma a_n^2$$

implies that of

$$\Sigma \left(\frac{A_n}{n}\right)^2.$$

It should be observed that Hilbert's theorem is a deeper and more difficult theorem than Theorems 1 and 2*.

* The simplest published proofs of Hilbert's Theorem are those given by Schur, "Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen" (*Journal für Math.*, vol. 140, pp. 1–28).

Theorems 3 and 4 are, as appears from the preceding analysis, substantially equivalent to Hilbert's. The converses of these theorems are untrue, as may be seen at once by supposing that a_n is 1 when n is a power of 2 and is zero otherwise.

I add a direct proof of Theorem 4. Suppose that the sequence $a_1, a_2, a_3, \ldots$, when arranged in descending order of magnitude, becomes $b_1, b_2, b_3, \ldots$. Then plainly

$$B_n = b_1 + b_2 + \ldots + b_n \geq A_n.$$

It is therefore sufficient to prove that

$$\Sigma \left(\frac{B_n}{n}\right)^2$$

is convergent: that is to say, we may, in proving Theorem 4, assume without real loss of generality that a_n decreases steadily as n increases. It is, moreover, in virtue of Theorem 2, enough to prove that

$$\Sigma \frac{a_n A_n}{n}$$

is convergent.

The last series may be written in the form

$$\sum_{m \leq n} \frac{a_m a_n}{n},$$

and will be convergent if

$$\sum_1^\infty S_k,$$

where

$$S_k = \sum_{k \leq n, m > k+1} \frac{a_m a_n}{n},$$

is convergent. But*

$$\sum_{k \leq n, m < k+1} \frac{a_m a_n}{n} \leq \frac{1}{k} \sum_{k \leq n, m < k+1} \frac{a_m a_n}{m}$$

$$= \frac{1}{k} \sum_1^\infty \frac{a_m}{m} \sum_{k \leq n, m < k+1} a_n \leq \frac{1}{k} \sum_1^\infty a_m a_{km}$$

$$\leq \frac{1}{k} \sqrt{\left(\sum_1^\infty a^2{}_m \sum_1^\infty a^2{}_{km} \right)} \leq \frac{1}{k\sqrt{k}} \sum_1^\infty a^2{}_m,$$

and so

$$\sum_1^\infty S_k \leq \sum_1^\infty \frac{1}{k\sqrt{k}} \sum_1^\infty a^2{}_m.$$

* Every inequality in the argument which follows is to be interpreted as meaning "the left-hand side is convergent, and less than or equal to the right-hand side, if the latter is convergent".

4. Theorems 1 and 2 may be generalised as follows:—

THEOREM 5. *If f and g are positive summable functions, and*

$$F(x) = \int_a^x f(t)\,dt, \quad G(x) = \int_a^x g(t)\,dt \quad (a > 0);$$

then the three hypotheses (1) *that*

$$\int_a^\infty \frac{fG}{x}\,dx, \quad \int_a^\infty \frac{Fg}{x}\,dx$$

are convergent, (2) *that*

$$\int_a^\infty \frac{FG}{x^2}\,dx$$

is convergent, and (3) *that*

$$\int_a^\infty \int_a^\infty \frac{f(x)\,g(y)}{x+y}\,dx\,dy$$

is convergent, are equivalent.

THEOREM 6. *If a_n and b_n are positive, and*

$$A_n = a_1 + a_2 + \ldots + a_n, \quad B_n = b_1 + b_2 + \ldots + b_n;$$

then the three hypotheses (1) *that*

$$\Sigma \frac{a_n B_n}{n}, \quad \Sigma \frac{A_n b_n}{n}$$

are convergent, (2) *that*

$$\Sigma \frac{A_n B_n}{n^2}$$

is convergent, and (3) *that*

$$\Sigma\Sigma \frac{a_m b_n}{m+n}$$

is convergent, are equivalent.

All the integrals and series in question are certainly convergent if

$$\int_a^\infty f^2\,dx, \quad \int_a^\infty g^2\,dx, \quad \Sigma a_n^2, \quad \Sigma b_n^2$$

are convergent.

ON CHARACTERISTIC LINES ON A SURFACE.

By *Prof. Harold Hilton* and *Miss R. E. Colomb.*

§ 1. A *characteristic line* on a surface is defined by the property that the tangent at any point P of the line is an equi-conjugate diameter of the indicatrix of the surface at P (the surface is supposed synclastic at P). These lines have some analogy with the asymptotic lines of the surface. They are real on those parts of the surface where the asymptotic lines are imaginary, and *vice versâ*.

The radius of curvature ρ of the geodesic touching a characteristic line at P is given by

$$1/\rho = \cos^2\theta/\alpha + \sin^2\theta/\beta,$$

where α, β are the principal radii of curvature of the surface at P, and 2θ is the angle between the characteristic lines through P. Since $\tan^2\theta = \beta/\alpha$ from the properties of the indicatrix, we have

$$\rho = \tfrac{1}{2}(\alpha+\beta) = H/2K,$$

where $$H = 1/\alpha + 1/\beta \text{ and } K = 1/\alpha\beta.^*$$

Hence the differential equation of the characteristic lines on a surface is

$$H(L\,du^2 + 2M\,du\,dv + N\,dv^2) = 2K(E\,du^2 + 2F\,du\,dv + G\,dv^2).$$

If the surface is referred to the characteristic lines as parametral curves

$$M = 0 \text{ and } L/E = N/G = 2K/H.\dagger$$

Since $$H(EG - F^2) = LG + NE - 2FM,$$

and $$K(EG - F^2) = LN - M^2,$$

$$L/E = N/G = \{(EG - F^2)\,K/EG\}^{\frac{1}{2}}.$$

§ 2. We have seen in § 1 that, when the surface is referred to its characteristic lines, L, M, N can be expressed in terms of E, F, G, and their derivatives with respect to u and v.

Hence no surface can be mapped on a distinct surface to

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* The notation used throughout is that of Forsyth's *Differential Geometry*; except that the parameters of any point on the surface are taken as u and v instead of p and q.

† See Eisenhart's *Differential Geometry*, p. 130; R. Hoppe, *Archiv Math. Phys.*, vol. lxix. (1883), p. 19; E. Pucci, *Atti della R. Accad. dei Lincei*, series IV. vol. v. (1889), p. 501; V. Reina, *Ibid.*, p. 881; T. Hayashi, *Science Reports Tôhoko Imperial Univ.*, series I. vol. iii. (1914), p. 217.

which it is applicable so that the characteristic lines on the two surfaces correspond.

We shall find the cases in which, when a surface is mapped on a sheet of its centro-surface (the locus of the principal centres of curvature), the characteristic lines on the given surface and on the sheet of the centro-surface correspond.

Refer the surface to its lines of curvature, so that $F=M=0$. Then (using the formulæ of Forsyth's *Differential Geometry*, § 81) the characteristic lines on the given surface are

$$\frac{E}{\beta}\,du^2-\frac{G}{\alpha}\,dv^2=0 \quad \text{.....(i)},$$

since $$E=L\beta, \quad G=N\alpha;$$

and on the two sheets of the centro-surface they are*

$$\{E\alpha^2\beta_2\,du^2+G\beta^2\alpha_2\,dv^2\}\{E\alpha^2\alpha_2^{\,2}\beta_2+G\alpha_1^{\,2}\alpha_2\beta^2+EG\alpha_2(\beta-\alpha)^2\}$$
$$+4EG\alpha^2\beta^2\alpha_1\alpha_2^{\,2}\beta_2\,du\,dv=0 \quad \text{.....(ii)},$$

and

$$\{E\alpha^2\beta_1\,du^2+G\beta^2\alpha_1\,dv^2\}\{E\alpha^2\beta_1\beta_2^{\,2}+G\beta^2\alpha_1\beta_1^{\,2}+EG\beta_1(\alpha-\beta)^2\}$$
$$+4EG\alpha^2\beta^2\alpha_1\beta_1^{\,2}\beta_2\,du\,dv=0 \quad \text{.....(iii)}.$$

If the equations (i) and (ii) coincide, we have, firstly,

$$2EG\alpha^2\beta^2\alpha_1\alpha_2^{\,2}\beta_2=0.$$

Now $\alpha_2=0$ makes the first sheet of the centro-surface degenerate, and $\beta_2=0$ makes it developable. Rejecting these two cases, we have $\alpha_1=0$, *i.e.*, $N/G=\phi(v)$.

Then, comparing the coefficients of du^2 and dv^2 in the same two equations, we get $\alpha\beta_2+\alpha_2\beta=0$, *i.e.*,

$$\alpha\beta=EG/LN=\{1/f(u)\}^2,$$

where f and ϕ are arbitrary functions.

Substituting

$$F=M=0, \quad L=Ef^2/\phi, \quad N=G\phi$$

in the Mainardi-Codazzi equations

$$\left.\begin{aligned}&2(EG-F^2)(L_2-M_1)+L(FG_1-GE_2)\\&\quad+M(GE_1-EG_1+2FE_2-2FF_1)\\&\quad+N(2EF_1-EE_2-FE_1)=0\\&2(EG-F^2)(N_1-M_2)+L(2GF_2-GG_1-FG_2)\\&\quad+M(EG_2-GE_2+2FG_1-2FF_2)\\&\quad+N(FE_2-EG_1)=0\end{aligned}\right\}\ \text{...(iv)},$$

* The suffixes 1 and 2 denote differentiation with respect to u and v respectively.

we get $$G_1 = 0, \quad E_2/E = 2f'^2\phi'/\phi(f'^2 - \phi^2),$$

so that G is a function of v, and E is the product of $\phi^2/(f'^2 - \phi^2)$ and a function of u.

Since u can be replaced by a function of u and v by a function of v, there is no loss of generality in supposing $G = 1$, and $E = \phi^2/(f'^2 - \phi^2)$. Then

$$E = \phi^2/(f'^2 - \phi^2), \quad F = 0, \quad G = 1,$$
$$L = f'^2\phi/(f'^2 - \phi^2), \quad M = 0, \quad N = \phi.$$

Now the equation giving the Gaussian curvature $\dfrac{LN - M^2}{EG - F^2}$ in terms of E, F, G, and their derivatives becomes, when $F = M = 0$ and $G = 1$,

$$4LNE = -2EE_{22} + E_2^2.$$

This gives f as a function of ϕ, ϕ', ϕ''. But f is a function of u only, and ϕ of v. Hence f is constant, and E, F, G, L, M, N are all functions of v only. This implies that the surface is a helicoid with $v = constant$ as helices.*

Now a real helicoid cannot have its helices as lines of curvature unless it is a surface of revolution. Hence:—

If the characteristic lines on a given surface and on one sheet of the centro-surface correspond, the surface is a surface of revolution with constant Gaussian curvature.

The characteristic lines on the given surface and on *both* sheets of the centro-surface cannot correspond (unless the surface is a sphere).

Suppose the characteristic lines on the two sheets correspond, but do not correspond to the characteristic lines of the given surface, then equations (ii) and (iii) coincide. This gives

$$\frac{\beta_2}{\beta_1} = \frac{\alpha_2}{\alpha_1} = \frac{\alpha_2^2\{E\alpha^2\beta_1\beta_2^2 + G\beta^2\alpha_1\beta_1^2 + EG\beta_1(\alpha - \beta)^2\}}{\beta_1^2\{E\alpha^2\alpha_2^2\beta_2 + G\alpha_1^2\alpha_2\beta^2 + EG\alpha_2(\beta - \alpha)^2\}},$$

which leads to $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$, rejecting, as before, the cases $\alpha_2 = 0$ and $\beta_1 = 0$. Therefore $\alpha - \beta$ is constant. Hence:—

If the characteristic lines on the two sheets of a given surface correspond, the difference of the principal radii of curvature of the given surface is constant.

The lines of curvature also correspond in this case.

§ 3. The reader will readily show that:—

When a surface is mapped on a parallel surface by the

* See Eisenhart, *Differential Geometry*, p. 183, Ex. 23.

*common normals, the characteristic lines cannot correspond, unless the surfaces are spheres or developable.**

This is included in the more general result:—

If two surfaces (not spheres or developable) correspond in such a way that normals at corresponding points are parallel, the characteristic lines cannot correspond unless the surfaces are similar and similarly situated.

For any two such surfaces the characteristic lines have the same Gaussian spherical representation, which we may suppose given. Confining our attention to any one of the surfaces, refer it to its characteristic lines as parametral curves.

Then

$$M=0, \quad L/E=N/G=\{(EG-F^2)\,K/EG\}^{\frac{1}{2}}=2K/H.$$

In general on the Gaussian sphere we have

$$ds^2=H(L\,du^2+2M\,du\,dv+N\,dv^2)-K(E\,du^2+2F\,du\,dv+G\,dv^2),$$

which becomes now

$$ds^2=K\,(E\,du^2+2F\,du\,dv+G\,dv^2).$$

Hence we may suppose KE, KF, KG given.

We may now use the result of Darboux *Théorie Générale des Surfaces*, vol. 2, §§ 429, 430; or proceed as follows:—

$$LK^{\frac{1}{2}}=\{(KE.KG-K^2F^2)\,KE/KG\}^{\frac{1}{2}},$$

and similarly $NK^{\frac{1}{2}}$, may be considered given.

Writing then

$$E=e^{2k}\mathbf{E}, \quad F=e^{2k}\mathbf{F}, \quad G=e^{2k}\mathbf{G}, \quad L=e^{k}\mathbf{L}, \quad M=0, \quad N=e^{k}\mathbf{N},$$

where $Ke^{2k}=1$, we may suppose $\mathbf{E}$, $\mathbf{F}$, $\mathbf{G}$, $\mathbf{L}$, $\mathbf{M}$, $\mathbf{N}$ given functions of u and v.

Substitute these values of E, F, G, L, M, N in the equations (iv) of § 2, and we get

$$2\,\mathbf{LF}(\mathbf{G}\,k_1-\mathbf{F}\,k_2)+2\,\mathbf{NE}\,(\mathbf{F}\,k_1-\mathbf{E}\,k_2)+2\,(\mathbf{EG}-\mathbf{F}^2)\,\mathbf{L}_2$$
$$+\mathbf{L}\,(\mathbf{FG}_1-\mathbf{GE}_2)+\mathbf{N}\,(2\,\mathbf{EF}_1-\mathbf{EE}_2-\mathbf{FE}_1)=0,$$

and a similar equation. These give k_1 and k_2 uniquely on the assumption that L, N, and $EG-F^2$ are not zero.

Thence K is obtained as a given function of u, v multiplied by an arbitrary constant of integration; and therefore E, F, G, L, M, N are given, except for a constant a multiplying L, M, N and the constant a^2 multiplying E, F, G.

* The "lines of torsion," *i.e.*, the lines bisecting the angles between the lines of curvature (Hayashi, *loc. cit.*) cannot correspond unless H is constant and the distance between the parallel surfaces is $2/H$.

Since a surface is given uniquely, except as regards position by E, F, G, L, M, N, the result readily follows.

Any arc of a characteristic curve is equal to the corresponding arc of its Gaussian spherical representation, if and only if $K=1$.

§ 4. We now proceed to find the surfaces on which the characteristic lines are geodesics.

Refer the surface to the characteristic lines. Since the parametral curves are characteristic,

$$M=0, \quad L/E=N/G=\{(EG-F^2)K/EG\}^{\frac{1}{2}}.$$

Since the parametral curves are geodesic,

$$\left.\begin{aligned} 2EF_1 &= EE_2+FE_1 \\ 2GF_2 &= GG_1+FG_2 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

The first of these equations combined with the first of § 2 (iv) gives

$$2G(EG-F^2)L_2+L(FGG_1-G^2E_2)=0.$$

Substituting for GG_1 from the second of equations (i) we have

$$2G(EG-F^2)L_2+L(2FGF_2-F^2G_2-G^2E_2)=0.$$

Therefore $$\frac{\partial}{\partial v}\left\{\frac{L^2G}{EG-F^2}\right\}=0 \quad \text{or} \quad \frac{\partial}{\partial v}(EK)=0.$$

Hence EK is a function of u only, and similarly GK is a function of v only.

There is then no loss of generality in supposing $EK=GK=1$, for we may replace u, v respectively by any functions of u, v without altering the parametral réseau.

Taking then $E=G=k^2$ $(=K^{-1})$, equations (i) become

$$\frac{\partial(F/k)}{\partial u}=\frac{\partial k}{\partial v}, \quad \frac{\partial(F/k)}{\partial v}=\frac{\partial k}{\partial u}.$$

This gives immediately

$$k=f(u+v)+\phi(u-v) \quad \text{and} \quad F/k=f(u+v)-\phi(u-v).$$

Therefore, putting f for $f(u+v)$, ϕ for $\phi(u-v)$,

$$\begin{aligned} &E=(f+\phi)^2, \quad F=f^2-\phi^2, \quad G=(f+\phi)^2, \\ &L=2(f\phi)^{\frac{1}{2}}, \quad M=0, \quad N=2(f\phi)^{\frac{1}{2}}. \end{aligned}$$

If we put $u+v=U$, $u-v=V$, we get, on taking $U=$ *constant*, $V=$ *constant* as parametral curves,

$$\left.\begin{aligned} &E=f(f+\phi), \quad F=0, \quad G=\phi(f+\phi) \\ &L=(f\phi)^{\frac{1}{2}}, \quad M=0, \quad N=(f\phi)^{\frac{1}{2}} \end{aligned}\right\};$$

where f denotes $f(U)$ and ϕ denotes $\phi(V)$. The surface is now referred to its lines of curvature.

Putting $f^{\frac{1}{2}}\,dU = du, \quad \phi^{\frac{1}{2}}\,dV = dv,$

we get the surface in the form

$$E = f+\phi, \quad F=0, \quad G=f+\phi, \quad L=(\phi/f)^{\frac{1}{2}}, \quad M=0, \quad N=(f/\phi)^{\frac{1}{2}};$$

where f is a function of u only and ϕ of v.*

The Gaussian expression for the curvature in terms of E, F, G, and their derivatives gives

$$4EGLN = -2EG(E_{22}+G_{11}) + EE_2G_2 + GE_1G_1 + GE_2^2 + EG_1^2,$$

when $F=M=0$.

Applying this to the present surface, we get

$$(f+\phi)(f''+\phi''+2) = f'^2+\phi'^2 \quad \ldots\ldots\ldots\ldots\text{(ii)}.$$

This functional equation may be solved as follows:—

Differentiate three times with respect to u; we can then eliminate ϕ, ϕ', ϕ'' and obtain

$$ff''f'''' + f'f''f'''' - ff'''f'''' - f''^2f''' = 0.$$

This gives, excluding for the moment $f''=0$,

$$ff'' - f'^2 = af + bu + c;$$

and similarly we have

$$\phi\phi'' - \phi'^2 = A\phi + Bv + C.$$

Substituting for f'^2 and ϕ'^2 in (ii) we have

$$f\phi'' + \phi f'' + (a+2)f + (A+2)\phi + bu + Bv + c + C = 0.$$

Therefore

$$\frac{\partial^2}{\partial u\,\partial v}(f\phi'' + f''\phi) = 0 \quad \text{or} \quad -f'''/f' = \phi'''/\phi'$$

(excluding the case $f'=\phi'=0$, which makes E and $G=0$).

Now $-f'''/f'$ is a function of u, and ϕ'''/ϕ' is a function of v. Hence they must both be a constant k (say). We deduce $k=0$, or else

$$f = -d\sin(k^{\frac{1}{2}}u+\lambda) + p, \quad \phi = D\cosh(k^{\frac{1}{2}}v+\Lambda) + P;$$

where d, λ, p, D, Λ, P are constants of integration.

Substituting for f and ϕ in (ii), we get, for all values of u and v,

$$k(d^2-D^2) - 2(p+P) \equiv d(kp+kP-2)\sin(k^{\frac{1}{2}}u+\lambda) + D(kp+kP+2)\cosh(k^{\frac{1}{2}}v+\Lambda);$$

* We make here an obvious change of notation: f and ϕ, u and v not having the same meanings as earlier in the section.

whence either

(1) $d=D=0$, $p+P=0$, *i.e.*, E and $G=0$, which may be excluded.

(2) $\quad d=0,\ k(p+P)=-2,\ D=\pm 2/k,$

or $\quad D=0,\ k(p+P)=2,\quad d=\pm 2/k.$

These cases are not essentially distinct. Taking the first case and substituting in (ii), we readily see that (with a change of notation in the constants) f and ϕ take the forms a and $\frac{1}{2}b\{\cosh(2v/b^{\frac{1}{2}}+2c)-1\}-a$, respectively. This gives, on taking new parameters r and θ defined by

$$ab\theta^2=(a+b)u^2,\quad r^2(a+b)=ab^2\sinh^2(v/b^{\frac{1}{2}}+c),$$

$$E=r^2,\ F=0,\ G=\frac{(a+b)^2r^2}{a\{ab^2+(a+b)r^2\}},\ L=\frac{b^{\frac{1}{2}}\{(a+b)r^2-a^2b\}^{\frac{1}{2}}}{(a+b)},$$

$$M=0,\quad N=\frac{ab^{\frac{3}{2}}(a+b)}{\{ab^2+(a+b)r^2\}\times\{(a+b)r^2-a^2b\}^{\frac{1}{2}}}.$$

These are the fundamental quantities of the surface of revolution $z=f(r)$, where

$$\{f'(r)\}^2=\frac{b\{(a+b)r^2-a^2b\}}{a\{(a+b)r^2+ab^2\}}=\frac{(m-1)mr^2-b^2}{mr^2+b^2},$$

writing $a+b=ma$.

(3) $k=0$ leads to

$$f=c,\quad \phi=v^2\pm 2(c+C)^{\frac{1}{2}}v+C,$$

or to $\quad f=u^2\pm 2(c+C)^{\frac{1}{2}}u+c,\quad \phi=C.$

These can be shown to give the surface just found in the particular case in which b is very large.

The case of $f''=0$, excluded above, will also be found to give the same result as in case (2). Hence:—

The only surfaces whose characteristic lines are all geodesic (excluding spheres and developables) are the surfaces of revolution $z=f(r)$, where

$$\{f'(r)\}^2=\frac{(m-1)mr^2-b^2}{mr^2+b^2}.$$

If σ is the arc of the meridian of such a surface, it will be found that

$$(\sigma+c)^2=mr^2+b^2.$$

§ 5. The characteristic lines cannot form an isometric réseau on any surface (other than the plane or sphere),

i.e., the surface cannot be divided up into small squares by characteristic lines.*

The question suggests itself whether a surface can be divided up by characteristic lines into small rhombuses with a constant angle other than a right angle.

In this case the surface referred to its characteristic lines has

$$E=\lambda,\quad F=k\lambda,\quad G=\lambda,\quad L=\mu,\quad M=0,\quad N=\mu,$$

where k is constant and λ, μ are functions of u, v.

Putting $u+v=U$, $u-v=V$, we get

$$E=(1+k)\,l,\quad F=0,\quad G=(1-k)\,l,\quad L=m,\quad M=0,\quad N=m,$$

where l, m are functions of U, V. The surface is now referred to its lines of curvature.

Codazzi's equations, § 2, (iv), become

$$(1-k)\,lm_2=ml_2,\quad (1+k)\,lm_1=ml_1;$$

the suffixes 1 and 2 denoting now differentiation with respect to U and V. Hence:—

$$l=f^{1+k}\phi^{1-k},\quad m=f\phi,$$

where f is a function of U, and ϕ of V.

The Gaussian equation for the curvature now becomes

$$2m^2=-(1-k)\,l_{11}-(1+k)\,l_{22}+(1-k)\,l_1^2/l+(1+k)\,l_2^2/l,$$

leading to $$\frac{2}{1-k^2}f^{1-k}\phi^{1+k}+\frac{d}{dU}\left(\frac{f'}{f}\right)+\frac{d}{dV}\left(\frac{\phi'}{\phi}\right)=0.$$

This is only possible if f or ϕ is constant, since f and ϕ are functions of distinct variables U and V respectively.

Suppose ϕ constant. Then E, F, G, L, M, N are functions of U only, and it will follow, as in § 2, that the surface is one of revolution.

Hence the only surfaces with the given property are the surfaces of revolution whose principal radii of curvature at every point have a constant ratio.

§ 6. The only surface whose intersections with two sets of mutually perpendicular parallel planes are characteristic lines is $\sec x/a.\sec y/a=e^{z/a}$.

In fact, $z=f(x,y)$ has $x=constant$ and $y=constant$ as characteristic lines if $r(1+q^2)=t(1+q^2)$; and $s=0$.

The latter equation gives $z=\phi(x)+\psi(y)$; and substituting this value of z in $r(1+q^2)=t(1+q^2)$, we get the result stated.

* The lines of torsion form an isometric réseau when H is constant.

Similarly, if the intersections with any two sets of parallel planes are characteristic lines, the surface is of the type

$$\sec\frac{\cos\alpha.x-\sin\alpha.y}{c}.\sec\frac{\cos\alpha.x+\sin\alpha.y}{c}=e^{\sin 2\alpha.z/c}.$$

There is no real surface (other than the plane and sphere) whose characteristic lines are its intersections with coaxial right circular cylinders and with planes through the axis; or whose characteristic lines are its intersections with planes through a line and with planes perpendicular to this line.

It will suffice to give the proof of the first theorem; the proof of the second theorem being somewhat similar.

Take the surface as $z=f(r,\theta)$, where

$$x=r\cos\theta,\quad y=r\sin\theta,\quad r=u,\quad \theta=v.$$

Then
$$E=1+f_1^2,\quad F=f_1f_2,\quad G=r^2+f_2^2,$$
$$L:M:N=rf_{11}:(rf_{12}-f_2):(rf_{22}+r^2f_1).$$

If the parametral curves are characteristic lines, so that $M=0$ and $LG=EN$,

$$rf_{12}=f_2 \text{ and } rf_{11}(r^2+f_2^2)=(1+f_1^2)(rf_{22}+r^2f_1).$$

The first equation gives

$$f(r,\theta)=\phi(r)+r.\psi(\theta);$$

and, substituting in the second, we obtain, after some reduction,

$$r\phi''(1+\psi'^2)=(\psi''+\psi+\phi')(1+\psi^2+2\psi\phi'+\phi'^2).$$

This holds for all values of r and θ.

Considering first $\phi'=0$, we see that $\dfrac{(\psi+\psi'')(1+\psi^2)}{(1+\psi'^2)}$ is a constant k; and then taking $r=0$, we have that $\psi+\psi''$ is a constant a, whence

$$\psi=b\cos(\theta+c)-a.$$

Substituting this value of ψ in

$$(\psi+\psi'')(1+\psi^2)=k(1+\psi'^2),$$

we have, for all values of $\cos(\theta+c)$,

$$a\{1+b^2\cos^2(\theta+c)-2ab\cos(\theta+c)+a^2\}$$
$$+k\{1+b^2-b^2\cos^2(\theta+c)\}=0,$$

so that *either* $\quad b=0,\quad k+a(1+a^2)=0,$

which leads eventually to

$$\psi=-a,\quad \phi=ar-(A^2-r^2)^{\frac{1}{2}}+B,$$

and the surface is the sphere

$$(z-B)^2+r^2=A^2,$$

or $a=k=0$, which gives

$$\psi=b\cos(\theta+c),\quad \phi=0,$$

and the surface is the plane

$$z=A+br\cos(\theta+c);$$

or if $b=0$, we have again the sphere.*

§ 7. On the surface of revolution $z=f(r)$ the characteristic lines are

$$\left(\frac{d\theta}{dr}\right)^2=\frac{f''}{rf'} \quad \text{.....(i)},$$

and can be found by one integration. We have found in § 4 the condition that they are geodesics. They may be shown to be geodesically parallel if

$$\{f'(r)\}^2=\frac{a^2e^{r^2/b^2}}{r^2-a^2e^{r^2/b^2}}.$$

The surfaces of revolution with *plane* characteristic lines may be found as follows:—

First consider the case in which the characteristic lines lie in planes parallel to the axis of revolution, then $r\cos\theta=a$ is a solution of (i), giving

$$\frac{a^2}{r^2(r^2-a^2)}=\frac{f''}{rf'},$$

whence $$f(r)=k\{(r^2-a^2)^{\frac{1}{2}}+a\sin^{-1}a/r\}+b.$$

Next consider the case in which the characteristic lines lie in planes making an angle $\cot^{-1}k$ with the axis (the angle is evidently constant), then $f=kr\cos\theta$ is a solution of (i), giving

$$rf''(k^2r^2-f^2)=f'(rf'-f)^2.$$

This may be integrated by putting $f=e^\rho u$, $r=e^\rho$, and we obtain

$$\frac{(f^2-r^2k^2)^{\frac{1}{2}}-c}{f}=\log\frac{B\{f+(f^2-r^2k^2)^{\frac{1}{2}}\}}{rk}$$

as the equation to determine $f(r)$.

§ 8. The most interesting property of characteristic lines on a conicoid seems to be the following:—

If a common tangent to two confocal conicoids Σ, Σ' *touches a characteristic line of* Σ, *then* Σ, Σ' *are harmonically conjugate with respect to the other two confocals through the point of*

* The corresponding results for the lines of torsion are immediate.

contact of Σ; i.e., the poles of any plane with respect to the four conicoids form a harmonic range.

Let the equations of Σ, Σ' be

$$x^2/\alpha+y^2/\beta+z^2/\gamma=1 \quad \text{and} \quad x^2/(\alpha+k)+y^2/(\beta+k)+z^2/(\gamma+k)=1.$$

Let (t_1, t_2, t_3) be the direction cosines of a common tangent, (n_1, n_2, n_3) the direction cosines of the normal at its point of contact (x, y, z) with Σ, and (b_1, b_2, b_3) those of the tangent to Σ at (x, y, z) perpendicular to the common tangent.

Then $$n_1=px/\alpha, \quad n_2=py/\beta, \quad n_3=pz/\gamma,$$

where $$p^{-2}=x^2/\alpha^2+y^2/\beta^2+z^2/\gamma^2.$$

Since the tangent plane at the point of contact of Σ' is perpendicular to that at (x, y, z), and passes through (x, y, z), its equation is

$$b_1X+b_2Y+b_3Z=b_1x+b_2y+b_3z;$$

X, Y, Z being current coordinates. Hence

$$(b_1x+b_2y+b_3z)^2=(\alpha+k)b_1^2+(\beta+k)b_2^2+(\gamma+k)b_3^2,$$

or $$k=(b_1x+b_2y+b_3z)^2-(\alpha b_1^2+\beta b_2^2+\gamma b_3^2),$$

i.e., $$k=(b_1x+b_2y+b_3z)^2-(\alpha b_1^2+\beta b_2^2+\gamma b_3^2)(x^2/\alpha+y^2/\beta+z^2/\gamma)$$

$$=-\alpha\beta\gamma\{\alpha^{-1}(yb_3/\beta-zb_2/\gamma)^2+\beta^{-1}(zb_1/\gamma+xb_3/\alpha)^2+\gamma^{-1}(xb_2/\alpha-yb_1/\beta)^2\}$$

$$=-\alpha\beta\gamma p^{-2}(t_1^2/\alpha+t_2^2/\beta+t_3^2/\gamma)$$

$$=-\alpha\beta\gamma/p^2D^2,$$

where D is the semi-diameter of Σ parallel to the common tangent.

If u, v are the parameters of the other two confocals through (x, y, z), we have

$$x=\left\{\frac{\alpha(\alpha+u)(\alpha+v)}{(\alpha-\beta)(\alpha-\gamma)}\right\}^{\frac{1}{2}}, \quad y=\left\{\frac{\beta(\beta+u)(\beta+v)}{(\beta-\gamma)(\beta-\alpha)}\right\}^{\frac{1}{2}},$$

$$z=\left\{\frac{\gamma(\gamma+u)(\gamma+v)}{(\gamma-\alpha)(\gamma-\beta)}\right\}^{\frac{1}{2}}.$$

Then $$r^2=x^2+y^2+z^2=\alpha+\beta+\gamma+u+v, \quad p^2=\alpha\beta\gamma/uv.$$

But $$r^2+2D^2=\alpha+\beta+\gamma,$$

so that $$D^2=-\tfrac{1}{2}(u+v).$$

Hence $$k=-\alpha\beta\gamma/p^2D^2=2uv/(u+v),$$

which is the harmonic mean between u and v, from which the result follows.

THE ELEVEN-TWENTYTWO SECTION OF THE ZETA FUNCTION.

By *J. R. Wilton, M.A., D.Sc.*

1. The present paper continues the investigation undertaken in two recent communications,* which will be referred to as the *Fourteen Section* and the *Eighteen Section* respectively.

Using the same notation as in the *Fourteen Section*, we require, in order to complete the solution of a dynamical problem, the value of the integral

$$I(v) = \int_0^x \left(\frac{Q}{D - x^2} - P \right) \frac{dx}{\sqrt{(X_1 X_2)}},$$

where in the case of the eleven-twentytwo section

$$I(v) = \frac{1}{11} \cos^{-1} \frac{R_0 + R_1 x + \ldots + R_9 x^9 + x^{10}}{(D - x^2)^{\frac{11}{2}}} \sqrt{(\tfrac{1}{2} X_1)} - \frac{\frac{1}{4}\pi}{11}$$

$$= \frac{1}{11} \sin^{-1} \frac{R_0 - R_1 x + \ldots - R_9 x^9 + x^{10}}{(D - x^2)^{\frac{11}{2}}} \sqrt{(\tfrac{1}{2} X_2)} - \frac{\frac{1}{4}\pi}{11},$$

and the leading coefficient in descending powers of x is

$$R_9 = 11P - 1 = N\{11P(v) - P(11v)\},$$

where the parameter v is equal to $K + fK'i$, and

$$f = \frac{2r+1}{22} \quad (r = 1, 2, \ldots, 10).$$

The final stage of the reduction for this section must await a further development of the theory, but the expression in terms of the variables a, b, and e, of which we have already made use, may be obtained, by means of ϵ and p, without difficulty, though the work is very heavy and somewhat uninteresting. We have to deduce

$$N\{11P(v) - P(11v)\} = N\frac{N_{11}}{\gamma_{11}}$$

[*Phil. Trans.*, p. 221, equation (17)] in terms of a, b, and e. We have

$$\gamma_{11} = [(y - x)x - y^3](y - x)^3 - xy(y - x - y^2)^3,$$

* "The Seven-Fourteen Section of the Zeta Function," p. 129 of the present volume; and "The Nine-Eighteen Section of the Zeta Function," *ibid.*, p. 149.

and, as in the fourteen section and in the eighteen section, we find

$$\gamma_{11} = \frac{\alpha^5 (m-m^2)^5}{32\,(\alpha-m)^{10}} \{\epsilon\,[(\epsilon-2)\,p^2+\epsilon]\,[(\epsilon^2-\epsilon-1)\,p^2+\epsilon] - p\,(p^2-\epsilon^2)\,(p^2-\epsilon^3+\epsilon^2-\epsilon)]$$

$$= -\frac{a-b}{b}\,\frac{\alpha^5 (m-m^2)^5}{32\,(\alpha-m)^{10}}\,\epsilon\,[(\epsilon-2)\,p^2+\epsilon]\,[(\epsilon^2-\epsilon-1)\,p^2+\epsilon],$$

by use of equation (1), p. 293, of *Phil. Trans.*

By a similar, though more tedious, reduction, we find

$$N_{11} = 3x^5 + 3x^4(-3y+y^2) + x^3(9y^2+2y^3+8y^4) + x^2(-3y^3-15y^4-9y^5+5y^6) + x\,(12y^5+3y^6-2y^7+y^8) - 2\,(y^6+y^7)$$

$$= 4\gamma_{11} + \frac{\alpha^5 (m-m^2)^5}{32\,(\alpha-m)^{10}}\,\epsilon\,(1+p)\,\{[\epsilon-p^2+\epsilon p\,(\epsilon-1)] \times [2p^2+2\epsilon^2-\epsilon\,(1+p)^2]+\epsilon^2\,(1-p)\,(1-p^2)\,[p-\epsilon\,(2+p)]\}.$$

And

$$\frac{N_{11}}{\gamma_{11}} = 4 + \frac{b\,(1+p)}{a-b}\left\{1 + \frac{2p\epsilon\,(\epsilon-1)\,(p^2-\epsilon)\,(p+\epsilon)}{[(\epsilon^2-\epsilon-1)\,p^2+\epsilon]\,[(\epsilon-2)\,p^2+\epsilon]}\right\}.$$

To complete the reduction, we find, by means of

$$b^4 = a^3 - a^3\,(1-a)\,e^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (8),$$

and the equations of *Phil. Trans.*, p. 285,

$$p = \frac{a-b^2}{b\,(1-a)}, \qquad \epsilon = \frac{a^2\,(1-a)\,[1+(1+2a)\,e^2]}{(1-b^2)\,(a^2+b^2)},$$

$$\epsilon - 1 = \frac{(1+a)\,(a+b^2)\,(a^2-b^2)}{a\,(1-b^2)\,(a^2+b^2)}, \qquad N = \frac{(1-a)\,(a-b)^3}{2a^3 e\,(1+b)},$$

$$N\,\frac{b\,(1+p)}{a-b} = \frac{(a-b)\,(a^2-b^2)\,(1-b)}{2a^3 e\,(1+b)},$$

$$p+\epsilon = \frac{a\,(a+b)\,\{-a\,[a-(1+a-a^2)\,e^2]+b\,(1-a^2e^2)\}}{b\,(1-b^2)\,(a^2+b^2)},$$

$$p^2-\epsilon = \frac{a\,(a^2-b^2)\,\{-a\,[a-(1+a-a^2)\,e^2]+b^2\,(1-a^2e^2)\}}{b^2\,(1-a)\,(1-b^2)\,(a^2+b^2)},$$

$$(\epsilon-2)\,p^2+\epsilon = \frac{(a^2-b^2)^3\,(1+b^2)}{ab^2\,(1-a)^2\,(1-b^2)\,(a^2+b^2)},$$

$$(\epsilon^2-\epsilon-1)\,p^2+\epsilon = \frac{a^2\,(a^2-b^2)^2\,\{a+(1+3a+5a^2+a^3)\,e^2+a^2\,(1+3a+a^2)\,e^4\}}{b^2\,(1-b^2)^2\,(a^2+b^2)^2},$$

$$a^4-b^4 = -a^3\,(1-a)\,(1-e^2), \qquad a^2-b^4 = a^2\,(1-a)\,(1+ae^2),$$

$$1-b^4 = (1-a)\,(1+a+a^2+a^3e^2).$$

Thus we have, after reduction,

$$N\{11P(v)-P(11v)\}=\frac{2(1-a)(a-b)^3}{a^3e(1+b)}+\frac{(1-b)(a-b)(a^2-b^2)}{2a^3e(1+b)}$$
$$-\frac{1+a}{a^2e}\,\frac{1-b}{1+b}\,\frac{AE(1+ae^2)[1+(1+2a)e^2]}{F(1-e^2)(1+a+a^2+a^3e^2)},$$

where

$$A=b(1-a^2e^2)-a[a-(1+a-a^2)e^2],$$
$$E=b^2(1-a^2e^2)-a[a-(1+a-a^2)e^2],$$
$$F=a+(1+3a+5a^2+a^3)e^2+a^2(1+3a+a^2)e^4.$$

Before we can proceed further, we must obtain the equation connecting a and e. To do this, we substitute

$$c=\frac{1-e}{(1+a)e}$$

in equation (5), p. 293, of *Phil. Trans.*

After reduction, we find

$$a^5e^5-a(1+2a)(1+a-a^2)e^4+a^2(1+a^2)e^3$$
$$-a(1-3a^2)e^2-(1+a+a^2)e+a^2=0\ldots(45).$$

By means of this equation we find

$$(1+a)(1+ae^2)[1+(1+2a)e^2]$$
$$=(1+a+a^2+a^3e^2)[1+(1+2a)e^2-e(1-a^2e^2)]\ldots(46),$$
$$a(1-e^2)F=e(1-a^2e^2)(1+ae)^2(1+a+a^2+a^3e^2)\ldots(47),$$
$$a[a-(1+a-a^2)e^2][1+(1+2a)e^2]$$
$$=e(1-a^2e^2)(1+a+a^2+a^3e^2)\ldots(48),$$
$$(1+a)(1+ae^2)[1+a^2-a(1+2a-a^2)e^2]$$
$$=(1+ae)(1-a^2e^2)(1+a+a^2+a^3e^2)\ldots(49),$$
$$(1+a)^2[1+(1+2a)e^2]$$
$$=(1+a+a^2+a^3e^2)\{(1+a-a^2)[1+(1+2a)e^2]+a^2e(1-a^2e^2)\}$$
$$=(1+a+a^2+a^3e^2)H,\text{ say}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(50).$$

Putting for the moment

$$1-a^2e^2=B,\quad a[a-(1+a-a^2)e^2]=C,$$

we find

$$\frac{1-b}{1+b}AE=\frac{1}{1+b}\{-2(B^2-C^2)+(1-a)(1+a+a^2+a^3e^2)B^2\}$$
$$-C(B+C)+B(B+C)(1-b+b^2);$$

and
$$B-C=(1-a^2)(1+ae^2),$$
$$B+C=1+a^2-a(1+2a-a^2)e^2.$$

Hence we find, after a straightforward reduction, in which use is made of equations (46)—(50),

$$K=-\frac{1+a}{a^2e}\,\frac{1-b}{1+b}\,\frac{AE(1+ae^2)[1+(1+2a)e^2]}{F(1-e^2)(1+a+a^2+a^3e^2)}$$
$$=\frac{(1-a)\{1+(1+2a)e^2-e(1-a^2e^2)\}}{ae^2(1+b)}$$
$$+\frac{1-ae}{ae}-\frac{(1-ae)H}{ae^2(1+a)^2}(1-b+b^2).$$

To complete the reduction, we require the relations

$$(1+a)^5=(1+a+a^2+a^3e^2)\{(1+2a)(1+a)^2-a^3H\},$$

easily derived from (45), and

$$(1+a)^3(1-a^2e^2)=(1+a+a^2+a^3e^2)\{(1+a)^2-a^2H\},$$
$$(1+a)^2\{1+(1+2a)e^2\}=(1+a+a^2+a^3e^2)H,$$

which are obvious transformations of the first relation. From these equations we readily find

$$\frac{2(1-a)(a-b)^3}{a^3e(1+b)}+\frac{(1-b)(a-b)(a^2-b^2)}{2a^3e(1+b)}$$
$$=-\frac{1}{ae}+\frac{11}{2a^2e}(a^2-ab+b^2)-\frac{3(1+2a)}{2a^3e}(1-b+b^2)$$
$$+\frac{(1-a)(1+2a)(1+a)^2}{a^3e(1+b)}+\frac{1}{2a^3e}(1-b+b^2-b^3).$$

Adding the value K, we have, after a slight reduction,

$$N\{11P(v)-P(11v)\}$$
$$=-1+\frac{11}{2a^2e}(a^2-ab+b^2)-\left\{\frac{3(1+2a)}{2a^3e}+\frac{(1-ae)H}{ae^2(1+a)^2}\right\}b^3,$$

where

$$H=(1+a-a^2)\{1+(1+2a)e^2\}+a^2e(1-a^2e^2)\ldots(51).$$

Numerical verification.

2. Although the simplicity of the result almost precludes the possibility of a mistake in the final expression obtained in the various cases for

$$N\{nP(v)-P(nv)\},$$

it is important to determine the particular region of the elliptic function with which we are dealing, and to settle certain other questions of a like nature; and this can often be done only by means of a numerical verification of the result given.

In the case of the fourteen, eighteen, and twentytwo sections, we must have* (with the notation of the quadric transformation in the *Fourteen Section*)

$$\pm \frac{2n}{\gamma'} \operatorname{zn} 2fG' = 2n\,NP(v) - n\,\frac{Q}{D}$$

$$= 2N\{nP(v) - P(nv)\} + 2 - n\,\frac{a^2+b^2}{a^2e},$$

where $n = 7, 9, 11$ in the respective cases.

To apply the test we make use of the relations

$$\frac{2ab}{a^2-b^2} = \frac{\gamma}{\gamma'} \operatorname{cn} 2fG', \qquad \frac{2b}{1-b^2} = \frac{\gamma}{\gamma'} \operatorname{cn} 2f'G',$$

$$\frac{2aeb}{a^2e^2-b^2} = \frac{\gamma}{\gamma'} \operatorname{cn} 2f''G',$$

where $$f, f', f'' = \frac{2r+1}{2n}, \frac{2r'+1}{2n}, \frac{2r''+1}{2n},$$

and r, r', r'' are properly chosen from the sequence 1, 3, 5, ... And, in order to simplify the calculation, we make

$$\gamma = \gamma' = \sin 45°.$$

The equations giving a, b, and e may be transformed into various other forms, some of which are more convenient than those given above. I have, however, thought it better to retain those forms which were actually used in the verification.

Thus, in the case of the fourteen section, a particular set of values is given by $14f$, $14f'$, $14f'' = 1, 3, 9$. With $\gamma = \gamma'$,

$$\frac{2ab}{a^2-b^2} = \operatorname{cn}\frac{G'}{7} = \cos 15°5'.7 = 0.96557 = \tan 43°59'.8,$$

$$\frac{2b}{1-b^2} = \operatorname{cn}\frac{3G'}{7} = \cos 43°26'.0 = 0.72617 = \tan 35°59'.1,$$

$$\frac{2aeb}{a^2e^2-b^2} = \operatorname{cn}\frac{9G'}{7} = -\cos 68°2'.3 = -0.37398 = -\tan 20°30'.3;$$

* During the preparation of the following numerical values of the functions involved in the foregoing investigation I have been in constant communication with Sir George Greenhill, and I owe very much to his continual advice and help.

whence
$$\frac{b}{a} = \tan 21°59'.9 = 0.40399,$$
$$b = \tan 17°59'.5 = 0.32477,$$
$$\frac{b}{ae} = -\tan 10°15'.1 = -0.18086.$$

The fourteen section.

We should have [after a slight transformation by means of equations (8) and (9)]

$$14\sqrt{2}\,\mathrm{zn}\,2fG' = -\frac{7b}{ae} + \frac{3}{ab} + \frac{7}{b} + \frac{2a}{b} + \frac{e}{b} + \frac{4ae}{b} + \frac{2a^2e}{b} = \theta, \text{ say.}$$

The following table gives the comparison of the two expressions:—

f, f', f''	1, 3, 9 / 14	3, 9, 27 / 14	9, 27, 81 / 14
b/ae	−0.18086	0.40399	0.32477
$1/ab$	3.8306	9.9289	−1.1082
$1/b$	3.0792	−5.5292	2.4753
a/b	2.4753	3.0793	−5.5292
e/b	−6.8783	−4.4450	−1.3785
ae/b	−5.5292	2.4753	3.0793
a^2e/b	−4.4450	−1.3785	−6.8783
θ	1.378	−2.888	−2.147
$14\sqrt{2}\,\mathrm{zn}\,2fG'$	1.366	2.882	−2.151

In this table, and in all of those given below, the value of the zn function is taken directly from the British Association Tables, 1912.

The eighteen section.

In this case we should have

$$\pm 18\sqrt{2}\,\mathrm{zn}\,2fG' = -\frac{9b}{ae} + 2b^3 - \frac{3b^3}{a^3e} - \frac{6b^3}{a^2e} = \theta, \text{ say.}$$

It happens that the explicit value of b can be found: it is

$$\pm b = 3^{\frac{1}{3}}(2^{\frac{1}{2}} - 3^{\frac{1}{2}})^{\frac{1}{2}} = 0.359388.$$

The comparison is given in the table:—

f, f', f''	$\frac{1, 3, 7}{18}$	$\frac{7, 21, 49}{18}$	$\frac{49, 147, 343}{18}$
b/ae	0.14265	−0.26742	−0.40802
$2b^3$	0.09285	−0.09285	0.09285
$-3b^3/a^3e$	−0.07124	0.01632	0.08753
$-6b^3/a^2e$	−0.12551	−0.08226	−0.23529
θ	−1.3877	2.2480	3.6173
$18\sqrt{2}\,\mathrm{zn}\,2fG'$	1.3870	2.2480	−3.6172
r	−0.03458	1.8918	−1.5285
$\sqrt{R}$	0.67998	−16.443	−2.4296

The twentytwo section.

In this case, if

$$u = \frac{2(1-ae)b^3H}{ae^2(1+a)^2},$$

where H is given by equation (51), we should have

$$\pm 22\sqrt{2}\,\mathrm{zn}\,2fG' = -\frac{11b}{ae} - \frac{3b^3}{a^3e} - \frac{6b^3}{a^2e} - u = \theta, \text{ say.}$$

We have, as in the previous cases, to choose values of f, f' and f'' such that equation (8) is satisfied, and also, for this section, equation (45). In the course of the work a severe test of the arithmetic is supplied by the relation

$$b^4H = a^3\{1 + (1+2a)e^2\} + a^2e(1-a^2e^2),$$

which is easily obtained from (45).

The following table gives the comparison for the five sections which can be obtained in this case.

The fifth figure in the value of a, b, or e is not to be depended upon, so that in every case the agreement is as close as could be expected.

f, f', f''	$\frac{1,\ 3,\ 9}{22}$	$\frac{5, 3.5, 9.5}{22}$	$\frac{5^2, 3.5^2, 9.5^2}{22}$	$\frac{5^3, 3.5^3, 9.5^3}{22}$	$\frac{5^4, 3.5^4, 9.5^4}{22}$
b	0.37723	−0.22508	−0.11744	−0.41006	−0.31402
a	0.91994	−0.71677	0.31132	−1.8218	−2.6738
e	3.4916	0.76579	−1.2012	−0.59666	0.52177
$-11b/ae$	−1.2918	−4.5107	−3.4542	4.1495	−2.4759
$-3b^3/a^3e$	−0.0592	−0.1213	−0.1341	0.0573	−0.0093
$-6b^3/a^2e$	−0.1090	0.1739	−0.0835	−0.2089	0.0498
$-u$	0.0614	−0.1080	0.0228	0.0032	0.1397
θ	−1.3986	−4.5661	−3.6490	4.0011	−2.2967
$22\sqrt{2}\,\mathrm{zn}\,2fG'$	1.3992	4.5656	3.6486	−4.0019	2.2966

[After the above calculations had been made Sir George Greenhill communicated to me a formula by means of which b/a, b, and b/ae might be calculated directly from the British Association tables* in terms of the functions $A(r)$, $B(r)$, $C(r)$ and $D(r)$. Putting

$$\tfrac{1}{2} - f = \frac{r}{90},$$

this formula (which does not determine the *sign* of the function) is

$$\frac{b}{a} = \frac{A(r)\,B(r)}{C(r)\,D(r)}, \quad \frac{b}{ae} = \pm\frac{A(2r)\,B(2r)}{C(2r)\,D(2r)}, \quad b = \pm\frac{A(3r)\,B(3r)}{C(3r)\,D(3r)}.$$

In conjunction with the values of a, b, and e given above these equations may be used to test the British Association tables.†]

* Section A (Dundee, 1912), *The further Tabulation of Bessel and other Functions*, p. 12.

† I have had, in the preparation of these tables of values of $a, b,$ and e, to make considerable use of the British Association table, on p. 12, for $\theta = 45°$. There can be no doubt that the arrangement according to equal increments of $F(\phi)$, instead of according to equal increments of ϕ, is of advantage, although it would have been a gain, in the particular calculations required, to have had one extra figure in the value of ϕ.

My impression is that the table still requires "polishing"—that the values of $A(r)$, $B(r)$, $C(r)$, and $D(r)$ are, in general, correct only to five significant figures.

The surd expressions given below for a, c, and A, in the case of the sixteen section, will be of use for testing the results obtained in this process of polishing.

The sixteen section.

The values of a and c in this case are given by the equations

$$\frac{a^2-1}{a^2+1}=\pm \operatorname{sn}\tfrac{1}{4}G' \text{ or } \pm \operatorname{sn}\tfrac{3}{4}G', \quad \frac{c^2-1}{c^2+1}=\pm \operatorname{sn}\tfrac{1}{2}G'.$$

When $\gamma=\gamma'=1/\sqrt{2}$ the exact expressions for c and a are

$$c,\ \frac{1}{c}=\pm\sqrt{(\sqrt{2}+1)}\pm\sqrt[4]{2},$$

$$a,\ \frac{1}{a}=\{1\pm\sqrt{(\sqrt{2}-\sqrt[4]{2})}\,\sqrt{[\sqrt{2}+1-\theta\sqrt{(\sqrt{2}+1)}]}\}$$
$$\times\sqrt{(\sqrt{2}+1)}\,\sqrt{(\sqrt[4]{2}+1)}\,\sqrt{\{1-\theta\sqrt{(2-\sqrt{2})}\}},$$

where $\theta=\pm 1$, according as the argument of the sn function is $\frac{1}{4}G'$ or $\frac{3}{4}G'$.

Taking $\gamma=\gamma'=1/\sqrt{2}$,

we have $NP(8v)=-\sqrt{2}$;

and, using the notation* of *Fourteen Section*, p. 137, we have the relation

$$A-\sqrt{2}=\frac{2a^2}{c}-\frac{4}{a}-\frac{3c^4+8c^3+2c^2-3}{2c^2}$$
$$=8\sqrt{2}\ (\mathrm{zn},\ \mathrm{zs})\ 2f'G',$$

where $f'=1-f$.

The results, which differ somewhat from those obtained for the previous sections, are exhibited in the following table:—

f	a	c	$A-\sqrt{2}$	$8\sqrt{2}\,\mathrm{zn}\,2f'G'$	$8\sqrt{2}\,\mathrm{zs}\,2f'G'$
$\frac{9}{16}$	5.94080	2.74298	0.58733	0.58728	
	−5.94080		1.9339		1.9342
$\frac{11}{16}$	1.60358	−0.364567	−6.4708		6.4721
	−1.60358		−1.4824	1.4818	
$\frac{13}{16}$	0.623605	−0.364567	1.5829	1.5827	
	−0.623605		14.412		14.415
$\frac{15}{16}$	0.168328	2.74298	−48.215		48.228
	−0.168328		−0.6890	0.6887	

* There is a misprint in the equation at the end of §3. This should read

$$N\{8P(v)-P(8v)\}=A+\frac{2\sqrt{B}}{ab}.$$

It may be noticed that in each case

$$\sqrt{2}\,\mathrm{zs}\,2f'G' - \sqrt{2}\,\mathrm{zn}\,2f'G' = 1/a.$$

The twenty section.

In this case the values of a and c are given by the equations

$$\pm\,\mathrm{cn}\,\frac{1,3}{5}\,G' = \frac{2c}{a+1/a}\,,\quad \pm\,\mathrm{cn}\,\frac{4,2}{5}\,G' = \frac{2}{a+1/a}\,.$$

And the relation we have to obtain is found to be

$$A \pm \sqrt{2} = 10\sqrt{2}\,(\mathrm{zn},\,\mathrm{zs})\,2f'G',$$

where

$$\gamma = \gamma' = 1/\sqrt{2},$$

$$A = K - \frac{5}{a} = \frac{a-1}{a^2c}\left\{1 + \frac{(a^2-c^2)\,(t-2y)}{ty\,(1-y^2)}\right\} - \frac{5}{a}\,,$$

$$f = \frac{11,\,13,\,17,\,19}{20}\,,$$

and, as in the sixteen section, $f' = 1 - f$.

The following table shows the connection between the functions:—

f	a	c	K	$A+\sqrt{2}$	$A-\sqrt{2}$	$10\sqrt{2}\,\mathrm{zn}\,2f'G'$	$10\sqrt{2}\,\mathrm{zs}\,2f'G'$
11/20	7.4996	3.5628	−0.1559	0.5916		0.5915	
		−3.5628	0.1559		−1.9250		1.9247
13/20	−2.1822	0.68706	5.3010		6.1781		6.1767
		−0.68706	−5.3010	−1.5955		1.5948	
17/20	−0.45826	0.68706	−14.080	−1.755		1.761	
		−0.68706	14.080		23.577		23.585
19/20	0.13334	3.5628	−36.799		−75.711		75.697
		−3.5628	36.799	0.715		0.697	

As the tables allow of an error in the value of a or c of about 1 in 5,000, the agreement is as close as could be expected.

Again, it will be observed that in each case

$$\sqrt{2}\,\mathrm{zs}\,2f'G' - \sqrt{2}\,\mathrm{zn}\,2f'G' = 1/a.$$

Table of values of γ_n.

3. I append a table of the values of $\gamma_1, \gamma_2, \ldots, \gamma_{14}$ in terms of ϵ and p. This table will be of considerable use in lessening the labour of the evaluation of $nP(v)-P(nv)$, for higher values of n, in terms of the same variables ϵ and p. It is to be used with the following sequence formulæ, easily derived from equation (7), p. 221, of *Phil. Trans.*, together with

$$\gamma_{2n+1}=\gamma_{n+2}\gamma_n^{\ 3}-\gamma_{n-1}\gamma^3_{n+1},$$
$$\gamma_{2n}=\gamma_n(\gamma_{n+2}\gamma^2_{n-1}-\gamma_{n-2}\gamma^2_{n+1}).$$

Let $$\psi_n=n\gamma_n\{nP(v)-P(nv)\}.$$

Then $$\psi_{2n+1}=\gamma_n^{\ 5}\psi_{n+2}-\gamma^3_{n+1}\psi_{n-1}+3(\gamma_{n+2}\gamma_n^{\ 2}\psi_n-\gamma_{n-1}\gamma^2_{n+1}\psi_{n+1}),$$

$$\psi_{2n}=-(n^2+1)(1+y)\gamma_{2n}+\gamma_n\gamma^3_{n-1}\psi_{n+2}-\gamma_n\gamma^2_{n+1}\psi_{n-2}$$
$$+(\gamma_{n+2}\gamma^2_{n-1}-\gamma_{n-2}\gamma^2_{n+1})\psi_n-2\gamma_n(\gamma_{n-2}\gamma_{n+1}\psi_{n+1}-\gamma_{n+2}\gamma_{n-1}\psi_{n-1}).$$

Putting $$k=\frac{\alpha(m-m^2)}{2(\alpha-m)^2},$$

$$x=-k\frac{(1-p)^2}{1+p},\quad y=-k\left[1+\epsilon-\frac{4p}{1+p}\right],\quad y^2=kp(1+\epsilon);$$

$$\gamma_1=1,$$
$$\gamma_3=x^{\frac{1}{2}},$$
$$\gamma_5=k(p-\epsilon),$$
$$\gamma_7=k^2\{-(p^2-\epsilon)+\epsilon(\epsilon-1)p\},$$
$$\gamma_9=x^{\frac{1}{2}}k^3\{\epsilon^2[(\epsilon-2)p^2+\epsilon]-p[p^2-\epsilon(\epsilon^2-\epsilon+1)]\},$$
$$\gamma_{11}=k^5\{\epsilon[(\epsilon-2)p^2+\epsilon][(\epsilon^2-\epsilon-1)p^2+\epsilon]$$
$$-p(p^2-\epsilon^2)[p^2-\epsilon(\epsilon^2-\epsilon+1)]\},$$
$$\gamma_{13}=k^7\{-\epsilon[(\epsilon-2)p^2+\epsilon]$$
$$\times[(\epsilon^3-2\epsilon^2-\epsilon+1)p^4+\epsilon(\epsilon^2+2\epsilon-1)p^2-\epsilon^4]$$
$$+p[p^2-\epsilon(\epsilon^2-\epsilon+1)][p^2-\epsilon-\epsilon(\epsilon-1)p][p^2-\epsilon+\epsilon(\epsilon-1)p]\};$$

$$\gamma_2=1,$$
$$\gamma_4=y=-k(1+\epsilon-3p+p\epsilon)/(1+p),$$
$$\gamma_6=-x^{\frac{1}{2}}k\epsilon(p+1),$$
$$\gamma_8=-k^3(1+\epsilon-3p+p\epsilon)(\epsilon-1)(p^2-\epsilon)/(1+p),$$
$$\gamma_{10}=k^4(p^2-\epsilon^2)[(\epsilon^2-\epsilon-1)p^2+\epsilon],$$
$$\gamma_{12}=x^{\frac{1}{2}}k^6(1+\epsilon-3p+p\epsilon)\epsilon[(\epsilon-2)p^2+\epsilon][p^2-\epsilon(\epsilon^2-\epsilon+1)],$$
$$\gamma_{14}=k^8[(p^2-\epsilon)^3-\epsilon^2(\epsilon-1)^2p^2]$$
$$\times[(\epsilon^3-2\epsilon^2-\epsilon+1)p^4+\epsilon(\epsilon^2+2\epsilon-1)p^2-\epsilon^4].$$

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CLASSIFICATION OF QUARTIC CURVES, MODULO 2.

By *Prof. L. E. Dickson.*

1. Let Q be the general homogeneous quartic function of x, y, z with integral coefficients. The points (x, y, z) for which the three first partial derivatives of Q are congruent to zero, modulo 2, are called the derived points of $Q \equiv 0$. Such a point is called a singular point or apex according as it is on or not on the locus $Q \equiv 0$. In the algebraic theory, Euler's relation excludes apices. But in the present number-theory case, $x^3y + z^4 \equiv 0 \pmod 2$ has the apices $(0a1)$, where a is arbitrary, and the one singular point (010).

Evidently the partial derivatives of Q do not involve the coefficients of the six terms $x^4, \ldots, y^2z^2$ containing only even exponents. Let E be the sum of these terms and $Q = E + O$. Thus the derived points depend only upon

$$O = dx^3y + exy^3 + fx^3z + gxz^3 + hy^3z + iyz^3 + mx^2yz + nxy^2z + pxyz^2.$$

First, let the first six coefficients of O be zero. We exclude the case in which also $m = n = p = 0$, since $Q = E = 0$ is then a conic counted twice. When x is replaced by $x + y$, O has the increment $(m + n)\, y^3z + px^2y^2$. We are thus led to the second case below, unless $m \equiv n \equiv p \pmod 2$, when O is

$$O_1 = x^2yz + xy^2z + xyz^2.$$

Second, let the first six coefficients of O be not all zero. Permuting the variables, we may set $d = 1$. Replacing x by $x + ey$, we have an O with $e = 0$. Replacing y by $y + fz$, we have also $f = 0$. Replacing x by $x + mz$, we have also $m = 0$. Thus

$$O' = x^3y + gxz^3 + hy^3z + iyz^3 + nxy^2z + pxyz^2.$$

2. Let $h = 0$ in O'. If $n = 0$, $g = i = 1$, interchange x and z, and then replace y by $y + z$, x by $x + pz$; we get O' with $h = n = g = 0$, $i = 1$; hence if $n = 0$, $i = 1$, we may set $g = 0$. The derived points of O', with $h = 0$, are $P = (010)$ and the points $(xy1)$, for which

$$x^3 + px + i = 0, \qquad ny^2 + x^2y + py + g = 0.$$

Listing together (by use of the same subscripts on O) the cases in which there is the same number of real and the same number of imaginary pairs of solutions x, y, we have

$O_2 = x^3y + xy^2z + xyz^2$: 00, 01, 10.

$O_3 = x^3y + xyz^2$: 00, $1a$ (a arbitrary).

$O_4 = x^3y$: $0a$ (a arbitrary).

$O_5 = x^3y + yz^3 + xy^2z$: 10, 11, $k0$, kk^2, $k^2 + k + 1 = 0$.

$O_6 = x^3y + xy^2z$: 00; $O_6' = x^3y + xz^3 + xy^2z$: 01;

$O_6'' = x^3y + xz^3 + xyz^2$: 01.

$O_7 = x^3y + yz^3$: 10, $k0$; $O_7' = x^3y + xz^3 + xy^2z + xyz^2$: 11, $0k$.

$O_8 = x^3y + yz^3 + xyz^2$: $t0$, $t^3 + t + 1 = 0$.

$O_9 = xy^2z + O_8$: $t0$, $tt^2 + 1$; $O_9' = O_9 + xz^3$: $tt + 1$, $tt^2 + t$.

$O_{10} = x^3y + xz^3 + yz^3 + xy^2z$: $1k$, kl, $l^2 + k^2l + 1 = 0$.

$O_{11} = x^3y + xz^3$: none.

The three O_6's are equivalent. Noting that the two derived points of each are on $x = 0$, we find that by replacing z by $y + z$ in O_6' we get O_6'', while O_6 is transformed into $O_6' + x^4$ by replacing y by $x + y + z$.

The two O_7's are equivalent. By considering the transformations replacing the line $y = 0$ of the imaginary derived points of O_7 by the corresponding line $x = 0$ for O_7', we are led to the transformation $x = X + Y$, $y = X$, $z = Y + Z$, which replaces O_7 by $\alpha + X^4 + X^2Y^2$, where α is O_7' written in capitals.

The two O_9's are equivalent. Considering transformations leaving fixed the single real derived point P of each, we find that the replacement of y by $x + y + z$ adds xz^3 to O_9.

No two unaccented O's are equivalent, since they do not have the same number of real derived points and the same number of imaginary ones in which the imaginaries are of like order.

3. Finally, for $h = 1$, we shall prove that only two new types O arise. Then (xzy) replaces O' by

$$O'' = x^3y + xz^3 + gy^3z + ixy^3 + nx^2yz + pxy^2z.$$

If $i = 0$, replace x by $x + nz$; we get $O' + nz^4 + pny^2z^2$, with $g' = 1$, $h' = g$, $i' = 0$, $n' = p$, $p' = n$. Hence, if $h = g = 1$, $i = 0$, we can interchange n and p; while if $h = 1$, $g = i = 0$,

we can set $h=0$. If $i=1$, replace x by $x+y+nz$; we get $O'+x^2y^2+nz^4+n(p+1)y^2z^2$, with $g'=1$, $h'=g+n+p$, $i''=1$, $n'=p$, $p'=n$. Hence, if $h=i=1$, we can set $h=0$ unless $g=1$, $n=p$; this is evident if $g=n+p$, while if $g=0$, $n+p=1$, we had $g'=1$, $h'=1$, $i''=1$, $n'+p'=1$, whence $g'=n'+p'$, so that we are led to the last case. The quartics not thereby proved equivalent to one with $h=0$ are those with $h=g=1$, $i=0$, $(n,p)=(1,1)$, $(0,0)$ or $(1,0)$, and those with $h=i=g=1$, $n=p$.

To the last with $p=1$ apply (xyz), then replace y by $y+z$ and x by $x+z$; we get x^3y+y^3z+E. Applying (xzy), we have an O' with $h=0$.

If $h=g=n=p=1$, $i=0$, we apply (xyz), then replace x by $x+z$ and get $O'+z^4$, with $h=g=i=1$, $n=p=0$, viz.,

$$O_{12}=x^3y+xz^3+y^3z+yz^3.$$

If $h=g=1$, $i=n=p=0$, $O'=x^3y+xz^3+y^3z$. Replacing x by $x+y$ and z by $z+y$, we get O'_9+E of § 2.

There remains only the case $h=g=n=1$, $i=p=0$, viz.,

$$O_{13}=x^3y+xz^3+y^3z+xy^2z.$$

The derived points of O_{12} and O_{13} are, respectively,

$$A=\left(\xi,\ \frac{1}{\xi^2},\ 1\right),\quad \xi^7+\xi^4+1=0;$$

$$B=\left(X,\ \ X+\frac{1}{X^2},\ 1\right),\quad X^7+X^6+1=0.$$

Since they are all imaginary, neither O_{12} nor O_{13} is equivalent to an $O+E$ of § 2. We next prove that the two are not equivalent to each other. The transformation

$$X=z,\quad Y=x+y,\quad Z=y,$$

replaces O_{12} by $O_{12}+E$. For $x=\xi$, $y=\xi^{-2}$, $z=1$, we have $X=1$, $Y=\xi^{-6}$, $Z=\xi^{-2}$. Hence the transformation replaces A by $(\xi^2,\ \xi^{-4},\ 1)$, another derived point of O_{12}. In fact, ξ^2 is a second root of the congruence in ξ, which is irreducible, modulo 2. Hence the powers of the transformation permute the seven derived points cyclically. If O_{12} and O_{13} are equivalent, there is therefore a real linear transformation replacing B by A, in which ξ and X are any chosen roots of the respective congruences of degree 7. Set $i=1/X$, so that $i^7+i+1\equiv 0$, the congruence used by W. H. Bussey* in

* *Bull. Amer. Math. Soc.*, vol. xii. (1905–6), p. 33. In the headings, $i^{-\lambda}$ is a misprint for i^{λ}.

constructing his table of the residues of powers of i. By means of that table we readily verify that $i^{29}=i^5+i$ is a value of ξ. The transformation (c_{ij}) replaces $B=(1,\ 1+i^3,\ i)$ by $(r,\ s,\ t)$, where

$$r=c_{11}+c_{12}(1+i^3)+c_{13}i,\quad t=c_{31}+c_{32}(1+i^3)+c_{33}i.$$

Since the new point is to be A, we have $r=\xi t$. Thus

$$(i^5+i)\,t=c_{33}i^6+(c_{31}+c_{22})\,i^5+c_{32}i^4+(c_{33}+c_{32})\,i^3+c_{31}i$$

must equal the cubic function r of i. But i^6, ..., i, 1 are linearly independent, modulo 2. Hence, $c_{33}=c_{32}=c_{31}=0$ and the determinant of (c_{ij}) is zero, contrary to hypothesis.

THEOREM. *Every quartic, modulo* 2, *is equivalent under real linear transformation to* $E+O$, *where* O *is one of the forms* $O_0\equiv O,\ O_1,\ \ldots,\ O_{13}$, *while* $E+O_i$ *and* $E+O_j$ *are not equivalent if* $i\neq j$.

The equivalence of two forms of the same system $E+O_j$ is answered readily by using the automorphs of that system, and they are easily found by using the fact that the derived points must be permuted amongst themselves.

4. In O_2 and O_3 we may delete the term x^3y by replacing z by $z+x$. From O_5 and O_7 we get x^3y+O_1 and $x^3y+x^2yz+xyz^2$, by replacing x by $x+z$. From O_{10} we get $x^3y+y^3z+yz^3+xy^2z$ by replacing x by $x+z$, y by x, z by $x+y+z$. To O_{11} we apply (xyz) and get x^3y+y^3z. The new forms and unchanged O_j were obtained in a quite different manner, independent of the present paper.

END OF VOL. XLIV.

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